A SECOND COURSE IN ELEMENTARY ALGEBRA

YOUNG AND JACKSON

TWENTIETH CENTURY TEXT-BOOKS

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PREFACE

The material in this volume is prepared for the use of pupils who have done a year's work in Elementary Algebra. Many high schools divide their Algebra work into two courses separated by some work in Geometry and elementary science. These schools often find it more convenient and economical to use two books in Algebra, one for the first course and another for the second. The authors' First Course in Elementary Algebra is planned for first-year work and this book is planned for the second-year work.

The first six chapters are a review and extension of the topics of the First Course. The chapter on logarithms is new in name only, because in theory it is an extension of the subject of exponents. The remainder of the volume treats the usual topics, Equations, Proportion, Variation and Series, supplemented by problems applying Algebra to Geometry.

Throughout the treatment the authors have constantly kept in mind both the logical value and the practical utility of the subject.

The logical value of Algebra is of prime importance; hence, the proofs of processes are based upon reasons both correct and satisfying to the mind of the pupil. On the other hand, subtle distinctions and arguments savoring of higher mathematical methods without their true rigor have been avoided.

The utility of Algebra is given the emphasis which it so richly deserves. This is done by making the equation prominent, by introducing simple formulas of Geometry and Physics, and by applying Algebra to modern industrial, commercial, and scientific problems whose content can readily be under-

stood by the pupil. Useless puzzles and problems relating to past conditions have been excluded, with the exception of a few supplementary problems retained on account of their historical interest.

The Summaries and Reviews at the ends of chapters furnish systematically, and in small compass, the essentials of Algebra. By reference to these the pupil can best review and unify his knowledge of the subject.

THE AUTHORS.

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A SECOND COURSE IN ELEMENTARY ALGEBRA

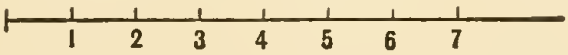
CHAPTER I

FUNDAMENTAL PROCESSES

1. Algebra is concerned with the study of numbers. The number of objects in any set (for example, the number of books on a shelf) is found by counting. Such numbers are called **whole numbers** or **integers**; also, **natural** or **absolute numbers.**

In arithmetic, numbers are usually represented by means of the numerals, 0, 1, 2, 3 . . . 9, according to a system known as the **decimal notation**, which we take for granted is here understood. In algebra, numbers are also represented by letters either singly or in combinations.

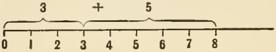
2. Graphical Representation. The natural integers may be represented by equidistant points of a straight line, thus:

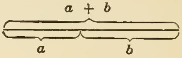


3. Addition. If two sets of objects are united into a single set (for example, the books on two shelves placed on a single shelf), the number of objects in the single set is called the sum of the numbers of objects in the two original sets. The process of finding the sum is called **addition**. The sign, +, between two number symbols indicates that the numbers are to be added. In the simplest instances the sum is found by counting.

Thus, to find $5+7$, we first count 5, and then count 7 more of the number words next following (six, seven, eight, etc.). The number word with which we end (twelve) names the sum.

4. Graphical Representation. The sum of two integers may be represented graphically thus:





Theoretically, the sum of two integers can in every instance be found by counting. But it is not necessary or desirable to do so when either (or both) of the numbers is larger than nine. In this case, the properties of the decimal notation, as learned in arithmetic, enable us to abridge the process of counting, and to find the sum very easily even when the numbers are large.

5. Commutative Law of Addition. If two sets of objects are to be united into a single set, the number of objects in the latter is obviously the same whether the objects of the second set are united with those of the first, or those of the first united with those of the second.

For example, the number of books is the same whether those on the first shelf be placed on the second, or those on the second be placed on the first. This is true because the operation of transfer neither supplies nor removes any books.

In symbols : $$a+b=b+a.$$

This fact is called the **commutative law of addition.**

The letters a and b are here used to stand for integers, but later they will be taken to stand for any algebraic numbers, and the law will still apply.

6. Graphical Representation. The commutative law may be represented graphically thus:

7. Addition of Two or More Integers. If more than two sets of objects are united into a single set, the number of objects in the resulting set is called the **sum** of the number of objects in the original sets, and the process of finding the sum is called **addition.** As in the case of two numbers, the sum of three or more numbers may be found by counting in the simplest instances, and for larger numbers, the process may be abridged by use of the properties of the decimal notation.

8. The commutative law likewise applies to the sum of three or more integers. That is:

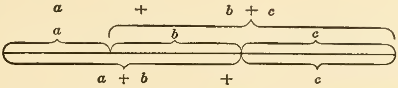
The sum is the same for every order of adding the numbers.

9. Associative Law of Addition. If we have three rows of books, the number of books is the same whether those in the second row are first placed with the first row, and then those in the third row placed with these, or those in the third row placed with the second, and then all of these with the first row.

In symbols: $(a+b)+c=a+(b+c)$.

This fact is called the **associative law of addition.**

10. Graphical Representation. The associative law may be represented graphically thus:



The properties stated above are often used to abridge calculations. Thus, $7+4+3+6$, are more easily added thus: $(7+3)+(4+6)$. In actual work the change of order is made merely by the eye.

ORAL EXERCISES

Rearrange advantageously and add:

1. $8+3+2+7$.
2. $91+43+9$.
3. $87+26+13$.
4. $48\,x+73\,x+2\,x+7\,x$.
5. $19\,y+54\,y+6\,y+y$.
6. $73\,b+186\,b+14\,b$.

7. $13a + 5a + 17a + 5a$.
8. $7x + 12x + 3x + 18x$.
9. $8y + 10y + 7y + 5y$.
10. $23a + 6a + 2a + 4a$.
11. $279t + 347t + 21t$.
12. $624p + 45p + 6p + 5p$.
13. $93t + 9t + 7t + t$.
14. $144m + 7m + 6m + 3m$.

WRITTEN EXERCISES

Show graphically that:

1. $$11 + 4 + 6 = 11 + (4 + 6).$$
2. $$8 + 5 = 5 + 8.$$
3. $$4a + 5b = 5b + 4a.$$
4. $$2a + 3a + 7a = 2a + (3a + 7a).$$

11. Subtraction. It often happens that we wish to know how many objects are left when some of a set are taken away, or to know how much greater one number is than another. The process of finding this number is called **subtraction.** The number taken away is called the **subtrahend**, that from which it is taken, the **minuend**, and the result, the **difference** or the **remainder.**

12. The sign of subtraction is $-$.

13. Subtraction is the reverse of addition, and from every sum one or more differences can at once be read.

Thus, from $5 + 7 = 12$ we read at once $12 - 5 = 7$,
$12 - 7 = 5$.

And from $5 + 5 = 10$, we read $10 - 5 = 5$.

And from $a + b = c$, we read $c - a = b$,
$c - b = a$.

Likewise, from $a + b + c = d$ we read $d - a = b + c$,
$d - (a + b) = c$, etc.

14. There is no commutative law of subtraction. For $7 - 4$ is not the same as $4 - 7$. In fact, the latter indicated difference has no meaning in arithmetic. We cannot take a larger number of objects from a smaller number.

15. In algebra, where numbers are often represented by letters, we may not know whether the minuend is larger than the subtrahend or not. For example, in $a - b$, we do not know whether a is larger than b or not. But it is desirable that such expressions should have a meaning in all cases, and this is accomplished by the definition and use of **relative numbers.**

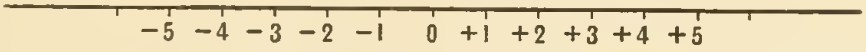
16. The First Extension of the Number System. Relative Numbers. Whenever quantities may be measured in one of two opposite senses such that a unit in one sense offsets a unit in the other sense, it is customary to call one of the senses the positive sense, and the other the negative sense, and numbers measuring changes in these senses are called **positive** and **negative** numbers respectively. (For examples, see FIRST COURSE, pp. 32–34.)

17. A number to be added is offset by an equal number to be subtracted; hence such numbers satisfy the above definition, and numbers to be added are called positive, and those to be subtracted are called negative. Consequently, positive and negative numbers are denoted by the signs $+$ and $-$ respectively.

Thus, $+5$ means positive five, and denotes five units to be added or to be taken in the positive sense.

-5 means negative five, and denotes five units to be subtracted, or to be taken in the negative sense.

18. Graphical Representation. Relative integers may be represented graphically thus:



It appears that the positive integers are represented by just the same set of points as the natural or absolute integers. For this and other reasons the absolute numbers are usually identified with positive numbers. Although it is usually convenient to do this, we have in fact the three classes of numbers: the absolute, the positive, and the negative. Thus, we may consider \$5 without reference to its relation to an account, or we can consider it as \$5 of assets, or we may consider it as \$5 of debts.

19. The following rules make clear in every instance whether the signs $+$, $-$ denote the operations of addition or subtraction, or the positive or negative character of the numbers which these signs precede:

I. *If used where a sign of operation is needed, the signs $+$, $-$, shall be regarded as signs of operation.*

For example:

In $8 - 5$, $-$ is a sign of operation (subtraction).

In $-8 + 5$, $-$ is a sign of character, because no sign of operation is needed before the 8, but $+$ is a sign of operation.

In the problem, "Add -8 and $+5$," both the signs are signs of character, because no sign of operation is needed; the operation has already been named.

II. *If it is necessary to distinguish a sign of character from a sign of operation, the former is put into a parenthesis with the number it affects.*

Thus, $-8 + (-3)$, means: negative 8 plus negative 3.

III. *When no sign of character is expressed, the sign plus is understood.*

Thus, $5 - 3$ means: positive 5 minus positive 3.

Similarly, $8a + 9a$ means: positive $8a$ plus positive $9a$.

20. Absolute Value. The value of a relative number apart from its sign is called its **absolute value.**

ORAL EXERCISES

Read the following in full, according to the agreements of Sec. 19:

1. $6 - 4$.	9. $12 - (-5)$.	17. $7 - 9$.
2. $-5 - 8$.	10. $-12 - (+5)$.	18. $7 + 9$.
3. $-8 + 20$.	11. $-7 - (-9)$.	19. $-7 + (-9)$.
4. $2a + 3a$.	12. $2a - (+3a)$.	20. $2y - (-3y)$.
5. $2b - 3b$.	13. $c + d$.	21. $-2x - (-3x)$.
6. $-2a - 3a$.	14. $c - d$.	22. $-2x + (-3x)$.
7. $2y + (+3y)$.	15. $m + (-n)$.	23. $-2b - (-5c)$.
8. $3p - (-2p)$.	16. $4x + (-2x)$.	24. $3a - (+5y)$.

WRITTEN EXERCISES

Indicate, using the signs $+$, $-$:

1. The sum of positive 5 and positive 3.
2. The sum of positive a and negative b.
3. The difference of positive p and positive q.
4. The difference of negative 5 and positive 3.
5. The difference of negative x and positive y.
6. The sum of positive a and positive b.
7. The sum of negative ab and negative ab.
8. The sum of positive y and negative x.
9. The difference of positive xy and negative xy.
10. The difference of negative pq and positive mn.

21. Addition of Relative Numbers. Just as 3 lb. + 5 lb. = 8 lb., so 3 positive units + 5 positive units make 8 positive units, and 3 negative units + 5 negative units make 8 negative units.

To add units of opposite character, use is made of the defining property of relative numbers, that a unit in one sense offsets a unit in the other sense. Thus, to add 3 positive units and 7 negative units we notice that the 3 positive units offset 3 of the negative units and the result of adding the two will be 4 negative units.

That is,

$$(+3)+(-7)=(+3)+(-3)+(-4)=-4.$$

In general:

I. *If two relative numbers have the same sign, the absolute value of the sum is the sum of the absolute values of the addends, and the sign of the sum is the common sign of the addends.*

II. *If two relative numbers have opposite signs, the absolute value of the sum is the difference of the absolute values of the addends, and the sign of the sum is the sign of the addend having the larger absolute value.*

22. More than two numbers are added by repetition of the process just described. This may be done either:

(1) *by adding the second number to the first; then the third number to the result, and so on;* or

(2) *by adding separately all the positive numbers and all the negative numbers, and then adding these two results.*

23. It may be verified that the Commutative and the Associative Laws of Addition hold also for relative integers.

24. Subtraction of Relative Numbers. Since n units of one sense are offset by adding n units of the opposite sense, we may subtract n units of one sense by adding n units of the opposite sense.

Thus, $7-(+3)=7+(-3)$.

And, $7-(-3)=7+(+3)$.

And, $4-(+7)=4+(-7)$.

25. Accordingly subtraction may be regarded as a variety of addition: *To subtract a monomial, we add its opposite.*

To subtract an algebraic expression consisting of more than one term, we subtract the terms one after another.

In general, *to subtract any algebraic expression we may change the sign of each of its terms and add the result to the minuend.*

The subtraction of a larger number from a smaller number (as in the third example, Sec. 24) is made possible by the introduction of the idea of relative numbers.

ORAL EXERCISES

State the sums:

1. $5+(-3)$.
2. $-6a+(-7a)$.
3. $-11y+3y$.
4. $-12z+(-18z)$.
5. $11x+(-2x)+(-5x)$.
6. $-3q+7q+(-6q)$.

7. How may the correctness of a result in subtraction be tested? State the differences:

8. $11-6$.
9. $-11-6$.
10. $-11a-(-6a)$.
11. $31x-(+5x)$.
12. $-31y-(-3y)$.
13. $17p-(-17p)$.

14. How may a parenthesis preceded by the sign + be removed without changing the value of the expression? One preceded by the sign − ?

15. How may terms be introduced in a parenthesis preceded by the sign + without changing the value of the expression? In a parenthesis preceded by the sign − ?

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} 2a+5 \\ a+4 \\ \hline \end{array}$

2. $\begin{array}{r} 3a+8 \\ a-4 \\ \hline \end{array}$

3. $\begin{array}{r} 6b+c \\ 3b-2c \\ \hline \end{array}$

4. $\begin{array}{r} -3a+b \\ 2a-3b \\ \hline \end{array}$

5. $\begin{array}{r} 4a-5 \\ 3a+7 \\ \hline \end{array}$

6. $\begin{array}{r} c+d-5 \\ c-d+5 \\ \hline \end{array}$

7. $\begin{array}{r} x+y+2z \\ x-y+4z \\ \hline \end{array}$

8. $\begin{array}{r} p+q-m \\ p-q+2m \\ \hline \end{array}$

9. $\begin{array}{r} 2x-y+z \\ 2x+2y-4z \\ \hline \end{array}$

10. $\begin{array}{r} ax+by+c \\ ax+y-c \\ \hline \end{array}$

11. $\begin{array}{r} 4x-2z+y \\ 2x-y+z \\ \hline \end{array}$

12. $\begin{array}{r} 1+m^3+p^2 \\ 1-m^3-p^2 \\ \hline \end{array}$

13. $\begin{array}{r} ax+by+cz^2 \\ bx+ay-z^2 \\ \hline \end{array}$

14. $\begin{array}{r} 1.5x+3.5y+z. \\ .5x+6.5y-.1z. \\ \hline \end{array}$

15. $\begin{array}{r} \frac{1}{2}x+\frac{4}{5}y-\frac{1}{3}z \\ \frac{1}{4}x-\frac{3}{5}y+\frac{2}{3}z \\ \hline \end{array}$

Subtract:

16. $\begin{array}{r} 4a+6 \\ 2a-9 \\ \hline \end{array}$

17. $\begin{array}{r} 8x+3 \\ -3x+2 \\ \hline \end{array}$

18. $\begin{array}{l} 11x-4y \\ 19x \\ \hline \end{array}$

19. $\begin{array}{r} 12t \\ -6t+3 \\ \hline \end{array}$

20. $\begin{array}{r} 7x+3y \\ 2y-4x \\ \hline \end{array}$

21. $\begin{array}{r} 4a+3b \\ 2c-5a \\ \hline \end{array}$

22. $\begin{array}{r} 16y+z \\ 8y-10z \\ \hline \end{array}$

23. $\begin{array}{r} 10x^2-16y^2 \\ 20x^2+4y^2 \\ \hline \end{array}$

24. $\begin{array}{r} 41x^2-16y^2 \\ 15x^2-20y^2 \\ \hline \end{array}$

25. $\begin{array}{r} -11m+40p \\ -40m-12p \\ \hline \end{array}$

26. $\begin{array}{r} x-7y+5z \\ -x+4y-6z \\ \hline \end{array}$

27. $\begin{array}{r} p+q-m \\ 6p-2q+4m \\ \hline \end{array}$

28. $\begin{array}{l} ax^2 + by^2 + c \\ \underline{ax^2 - by^2 - c} \end{array}$

29. $\begin{array}{r} m^2 + 2p^2 - 6q^2 \\ \underline{-m^2 - 4p^2 + 5q^2} \end{array}$

30. $\begin{array}{l} 40x^2 - 10y^2 + z^2 \\ \underline{50x^2 + 40y^2 - 7z^2} \end{array}$

31. $\begin{array}{l} 2.5a + 6.3b - .1c \\ \underline{1.5a - 3.5b - .9c} \end{array}$

32. $\begin{array}{l} \frac{1}{2}x - \frac{2}{3}y + \frac{1}{5}z \\ \underline{\frac{3}{4}x - \frac{1}{3}y + \frac{4}{5}z} \end{array}$

33. $\begin{array}{l} a^2x + b^2y + c^2z^2 \\ \underline{a^2x + by^2 + cz^2} \end{array}$

Remove parentheses and unite terms as much as possible:

34. $3a - 2b + (3b - 7a)$.

35. $4m + 5 - (63 - 3p)$.

36. $(11x + 5y) - (-3x + 2z)$.

37. $7 - [2 - (3 - 5)]$.

38. $(4a + 2a) - [(7a - 5a) + (-6a - 17a)]$.

39. $3 + \{5x - 2 - (7x - 1 - 2 + 3x)\}$.

40. $4x + \{2x - [3 - (7x + 5) - 1 + x]\}$.

41. $ab - \{2ab + c - [3c - (6 - ab)]\}$.

42. $12xy - \{2xy - 6z + [4xy - (2z + xy)]\}$.

43. $-(5pq - 3xy + 1) - \{-2pq + 3xy - (xy + 2)\}$.

44. $-\{4abc - (2ac + bc)\} + \{6abc + (2ac - bc)\}$.

Write the expression of Exercises 45–56, as x minus a parenthesis. Also group the terms involving x and y in each expression in a parenthesis preceded by the sign $-$.

45. $3a + x + 2y$.

46. $-5y + 7c - 8 + x$.

47. $6t^2 + 4y + x - 3$.

48. $2m + p + x - y$.

49. $a + x - b + cy$.

50. $p + x - y + by$.

51. $x - 3b + 2y - c$.

52. $3b + x + ay + by$.

53. $by + m - n + x$.

54. $2p - q + x - y$.

55. $ay + x - z + zy$.

56. $c + d + x - (a + b)y$.

26. Multiplication. To multiply two absolute integers means to use the one (called the **multiplicand**) as addend as many times as there are units in the other (called the **multiplier**). The result is called the **product.**

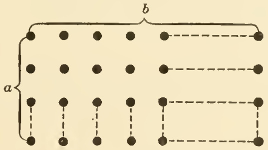
Thus, 3 times 4 means $4 + 4 + 4$.

The simpler products are obtained by actually making the additions that are implied. For large numbers, as we have seen in arithmetic, the process may be much abridged by use of properties of the decimal notation.

27. Commutative Law of Multiplication. The expression 3 times 5 means $5 + 5 + 5$, and may be indicated as follows: That is, since each horizontal line (or *row*) contains 5 dots, there are all together 3 times 5 dots. But each vertical line (or *column*) contains 3 dots and there are 5 columns.

(*A*) 

Hence there are 5 times 3 dots all together. But the number of dots is the same whether we count them by rows or by columns, hence *5 times 3 equals 3 times 5.* Quite similarly, if we have

(*B*) 

a rows of dots with b dots in each row, it follows that a times b equals b times a.

The result may be stated in symbols thus:

$$ab = ba.$$

This fact, called the **Commutative Law of Multiplication**, means that the product is not altered if multiplier and multiplicand are interchanged. Consequently, these names are frequently replaced by the name **factor** applied to each of the numbers multiplied.

28. Let each dot of the block of dots of (A) above have the value of 6. Since there are 3×5 dots, the value of the block would be $(3 \times 5) \times 6$.

A second expression for the value of the block is obtained by finding the value of the first row (viz. 5×6) and multiplying it by the number of rows, or 3. The expression resulting must be equal to that already found, or:

$$3 \times (5 \times 6) = (3 \times 5) \times 6.$$

Similarly, if each dot of the block (B) above has the value c, it follows that

$$a(bc) = (ab)c.$$

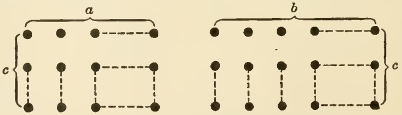
That is: *The product of three absolute integers is not altered if they be grouped for multiplication in any way possible without changing the order. This is a case of what is known as the* **Associative Law of Multiplication** *for absolute integers.*

29. By similar methods and use of these results it can be proved that both the Commutative Law and the Associative Law apply to all products of absolute integers. That is:

Commutative Law. *The product of any number of given factors is not changed, if the order of the factors be changed in any way.*

Associative Law. *The product of any number of given factors in a given order is not changed if the factors be grouped in any way.*

The Distributive Law. From the block of dots we see that



$$c(a+b) = ca + cb.$$

This is called the **Distributive Law** of Multiplication, and the above proof covers the case in which a, b, and c are absolute integers.

30. Multiplication of Relative Integers. To multiply by a positive integer means to take the multiplicand as addend as many times as there are units in the multiplier, and to multiply by a negative integer means to take the multiplicand as subtrahend as many times as there are units in the multiplier.

Consequently,

$$4 \cdot 3 = 3 + 3 + 3 + 3 = 12.$$
$$4(-3) = -3 + (-3) + (-3) + (-3) = -12.$$
$$(-4)3 = -3 - 3 - 3 - 3 = -12.$$
$$(-4)(-3) = -(-3) - (-3) - (-3) = 12.$$

So, generally,

$$(+a)(+b) = +ab,$$
$$(+a)(-b) = -ab,$$
$$(-a)(+b) = -ab,$$
$$(-a)(-b) = +ab.$$

31. In words: *The product of two (integral) factors of like signs is positive, and of two factors of unlike signs is negative; in each case the absolute value of the product is the product of the absolute values of the factors.*

32. We observe that the Commutative Law holds in this case also.

For example: $(-b)\,a = a\,(-b)$. Since $ba = ab$, by the Commutative Law for absolute integers, and by Sec. 30, $(-b)a = -ba = -ab = a(-b)$.

It may be shown that the Associative and the Distributive Laws also hold.

ORAL EXERCISES

State the laws that are applied in the various steps of the following calculations:

1. $2 \cdot 8 \cdot 3 \cdot 5 = 2 \cdot 5 \cdot 8 \cdot 3 = 10 \cdot 24 = 240.$
2. $5(17 \cdot 2 - 6\,c) = 5(17 \cdot 2) - 5(6\,c) = (5 \cdot 2)17 - (5 \cdot 6)\,c$
$= 10 \cdot 17 - 30\,c = 170 - 30\,c.$
3. $7\,b(5\,x + ab) = (7\,b)(5\,x) + (7\,b)(ab) = 35\,bx + 7\,ab^2.$
4. $(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$

State the products:

5. $\begin{array}{r} ax \\ \underline{5b} \end{array}$	7. $\begin{array}{r} 8x \\ \underline{-4xy} \end{array}$	9. $\begin{array}{r} 5x^2 \\ \underline{2x^2} \end{array}$	11. $\begin{array}{r} 6xyz \\ \underline{-2xz^3} \end{array}$
6. $\begin{array}{r} -2a \\ \underline{-3a^2} \end{array}$	8. $\begin{array}{r} -a^5b \\ \underline{a^2c} \end{array}$	10. $\begin{array}{r} -x^2y \\ \underline{-xy^2} \end{array}$	12. $\begin{array}{r} -4a^2t \\ \underline{-3b^2t} \end{array}$

13. $(a+b)^2$.
14. $(a-c)^2$.
15. $(x+y)^2$.
16. $(x-a)^2$.
17. $(x-2y)^2$.
18. $(x+y)(x-y)$.
19. $(2x-1)^2$.
20. $(2m+n)^2$.
21. $(x-1)(x+1)$.
22. $(1-y)(1+y)$.
23. $(m+2x)^2$.
24. $(3-2d)(3+2d)$.
25. $(4-x)^2$.
26. $-3x(x+2)$.
27. $(1-2d)(1+2d)$.

WRITTEN EXERCISES

Multiply and test (see First Course, p. 69):

1. $\begin{array}{l} 2a+3 \\ \underline{2a+4} \end{array}$	3. $\begin{array}{l} x+2 \\ \underline{x-3} \end{array}$	5. $\begin{array}{l} y^2-3y+4 \\ \underline{y-2} \end{array}$
2. $\begin{array}{l} 3a-2b \\ \underline{2a-3b} \end{array}$	4. $\begin{array}{l} x^2-3x+1 \\ \underline{x-2} \end{array}$	6. $\begin{array}{l} p-3t+t^3 \\ \underline{p-2t} \end{array}$

7. $(3x-4y)^2$.
8. $(5y-7t)(8y+2t)$.
9. $(6a+13q)^2$.
10. $(1+x)(2+x)(1-x)$.
11. $(1-2y)^2$.
12. $(b-3)(b+7)(b-3)$.

33. Division. Division is the process of finding a number called the **quotient**, such that when multiplied by a given number, called the **divisor**, the product is a given number, called the **dividend**.

34. The fundamental relation of division is:

Divisor times Quotient equals Dividend.

From this it follows at once that if dividend and divisor have the same sign, the quotient is positive, and if they have unlike signs, the quotient is negative.

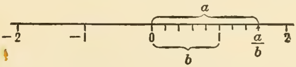
Division is usually indicated in one of the following forms: $\frac{8}{2}$ or $8 \div 2$. Both are read: "eight divided by two."

35. The Second Extension of the Number System; Fractions. We found the natural or absolute integers by counting. We defined the operation of addition for these integers and saw that it was always possible. Next we defined the operation of subtraction for these integers, and found that it was not always possible. This led us to define relative numbers (positive and negative). In the system of numbers as enlarged by this first extension, we saw that both addition and subtraction are always possible. Then we defined multiplication for all integers and saw that it was always possible. We now examine the operation of division as just defined.

The operation $12 \div 4$ is possible in the system of integers because there exists an integer, 3, whose product with 4 is 12. But the operation $12 \div 5$ is impossible in the system of integers, since there exists no integer whose product with 5 is 12. This leads us to define another kind of number, the fraction. This is done by dividing the unit into b equal parts, and taking a of these parts. A symbol for the new number is $\frac{a}{b}$, in which a and b may be any integers. This constitutes our **second extension** of the number system.

36. The new number, $\frac{a}{b}$, is called a **fraction**; a is called the **numerator** and b the **denominator** of the fraction; a and b together are called the **terms** of the fraction.

37. Graphical Representation of Fractions. Fractions may be represented by points of the number scale that we have already had.



Thus, if the distance from *zero* to 1 is divided into b equal parts, and then a of these parts laid off from *zero*, the end point represents the fraction $\frac{a}{b}$. $\left(\text{In the figure the fraction is } \frac{8}{5}.\right)$ Fractions may also be taken in the negative sense. Two fractions are said to be equal, or to have the same value, when they are represented by the same point of the number scale.

38. If each of the b equal parts is halved, making $2b$ parts, and each of the a parts taken is halved, making $2a$ parts, the end point remains the same.

That is:

$$\frac{a}{b}=\frac{2a}{2b}.$$

Similarly:

$$\frac{a}{b}=\frac{na}{nb}.$$

(A) In words: *The value of a fraction is not altered if both numerator and denominator is multiplied by the same integer.*

(B) Reading the above equation from right to left. *The value of a fraction is not altered if both numerator and denominator be divided by a common integral factor.*

Every integer may be regarded as a fraction.

Thus, $3=\frac{3}{1}$ or $\frac{6}{2}$ or $\frac{15}{5}$, etc.

39. These properties enable us to add and subtract fractions.

If the given fractions have the same denominator, we say 8 fifths plus 3 fifths are 11 fifths (or $\frac{8}{5}+\frac{3}{5}=\frac{11}{5}$) just as we say 8 ft. + 3 ft. = 11 ft.

If they have not the same denominator, they are first reduced to the same denominator, and then the results added.

40. To multiply a fraction by an integer we extend the definition of multiplication to this case, and we have:

$$3(\tfrac{4}{5})=\tfrac{4}{5}+\tfrac{4}{5}+\tfrac{4}{5}=\tfrac{12}{5}.$$

$$c\left(\frac{a}{b}\right)=\frac{a}{b}+\frac{a}{b}+\cdots\text{ to }(c\text{ terms})=\frac{a+a+a\cdots\text{ to }(c\ a\text{'s})}{b}=\frac{ca}{b}.$$

$$(-c)\left(\frac{a}{b}\right)=-\frac{a}{b}-\frac{a}{b}\cdots\text{ to }(c\text{ terms})$$

$$=\frac{-a-a-a\cdots\text{ to }(c\text{ terms})}{b}=-\frac{ca}{b}.$$

41. That is, *to multiply a fraction by an integer we multiply the numerator by that integer.*

The fraction $\frac{a}{b}$ is the quotient called for in the indicated division $a \div b$, because the product of the divisor b and the asserted quotient $\frac{a}{b}$ is the dividend a, as seen in

$$b \cdot \frac{a}{b} = \frac{ba}{b} = a \text{ (by property B, Sec. 38).}$$

In our present number system the fraction $\frac{a}{b}$ may therefore always be regarded as indicating the division of a by b (provided b is not zero).

42. The operation of division is thus seen to be possible in the new number system for any integral dividend and any integral divisor (except zero).

We shall see, under Division of Fractions, that if either dividend or divisor or both are fractions, the operation is still possible without enlarging the number system further.

43. Multiplication of Fractions. *The product of two or more fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

44. Since an integer or an integral expression may be regarded as a fraction with denominator 1, this definition applies also when one or more of the factors are integral.

45. The Associative, Commutative, and Distributive Laws of Multiplication may be shown to apply to fractions as well as to integers.

WRITTEN EXERCISES

Perform the indicated operations, and express the result in lowest terms:

1. $\frac{3}{5}+\frac{1}{10a}$.

2. $\frac{4a}{5}-\frac{5a}{4}$.

3. $\frac{3}{b-1}-\frac{2}{b+1}$.

4. $\frac{x-1}{x+1}-\frac{x+4}{2(x+1)}$.

5. $\dfrac{x^2}{2y}+\dfrac{3y}{4x}.$

6. $\dfrac{5}{a-b}+\dfrac{2}{b}.$

7. $\dfrac{2}{a+p}-\dfrac{3p}{(a+p)^2}.$

8. $\dfrac{5}{x^4}+\dfrac{3}{2x^2}-1.$

9. $\dfrac{2q}{1-q^2}+\dfrac{5}{1+q}.$

10. $\dfrac{x}{(1-x)(1-y)}+\dfrac{1}{(y-x)(y-1)}+\dfrac{y}{(x-1)(1-y)}.$

11. $\dfrac{1}{a^2-(b+c)a+bc}+\dfrac{1}{a^2-(b+d)a+bd}+\dfrac{1}{a^2-(c+d)a+cd}.$

12. $\dfrac{2xy+y^2}{(x-y)^2}-\dfrac{y}{x-y}-\dfrac{x^2+5xy}{(x+y)^2}.$

13. $\dfrac{5}{7a}\cdot\dfrac{14}{15b}.$

14. $\dfrac{x^3}{y^2}\cdot\dfrac{3xy}{4z^5}.$

15. $\dfrac{t^2-4}{(1+q)^2}\cdot\dfrac{q^2-1}{(2+t)^2}.$

16. $\dfrac{5x}{3y}\cdot\dfrac{9t^2}{16x^3y}\cdot\dfrac{4yz}{15t^3x}.$

17. $bc\cdot\dfrac{b}{c^2}\cdot\dfrac{ac^4}{2b^3}.$

18. $\dfrac{-3a}{x^2-9}\cdot\dfrac{4b}{9x}.$

19. $\left(\dfrac{3}{a}-\dfrac{2}{b}\right)^2.$

20. $x^3\left(1-\dfrac{1}{x}\right)^2.$

21. $pqr\left(\dfrac{p}{q}-\dfrac{q}{r}\right)^2.$

22. $\dfrac{a(1-a)}{1+2a+a^2}\cdot\dfrac{1+a}{1-2a+a^2}.$

23. $\dfrac{1-a^2}{1+2b+b^2}\cdot\dfrac{1-b^2}{b^2-2ab+a^2}\left(\dfrac{a}{1-a}-\dfrac{b}{1-b}\right).$

24. $\left(\dfrac{t+1}{t-1}-\dfrac{t-1}{t+1}-\dfrac{4t^2}{1-t^2}\right)\dfrac{t^2+2t+1}{4t}.$

25. $\left(\dfrac{v^2}{p^2}+\dfrac{p^2}{v^2}-\dfrac{v}{p}-\dfrac{p}{v}+1\right)\left(\dfrac{v}{p}-\dfrac{p}{v}\right).$

46. Reciprocals. If the product of two numbers is 1, each is called the **reciprocal** of the other.

Thus, the reciprocal of $\dfrac{a}{b}$ is $\dfrac{b}{a}$, since $\dfrac{a}{b}\cdot\dfrac{b}{a}=\dfrac{ab}{ab}=1.$

47. Division of Fractions. To divide $\frac{a}{b}$ by $\frac{c}{d}$ means to find a number q such that $\frac{c}{d} \cdot q = \frac{a}{b}$.

Multiplying both members by $\frac{d}{c}$, the reciprocal of $\frac{c}{d}$, we obtain

$$q = \frac{d}{c} \cdot \frac{a}{b} = \frac{ad}{bc}.$$

That is: *To obtain the quotient, multiply the dividend by the reciprocal of the divisor.*

We have thus seen that in the number system as now extended the four fundamental operations are possible when any or all of the numbers involved are fractions. (Division by zero is always excepted.)

48. Complex Fractions. If one or both of the terms of a fraction are themselves fractions, the given fraction is called a **complex fraction.** Since a fraction indicates division, a complex fraction may be simplified by performing the indicated division of its numerator by its denominator.

WRITTEN EXERCISES

Divide, and express results in lowest terms:

1. $\frac{x}{2} \div \frac{y}{6}$.

2. $\frac{ab}{18c} \div \frac{4b^2}{3ac}$.

3. $\frac{ax^2}{2y^3} \div \frac{bx^3}{ay^2}$.

4. $\dfrac{\frac{5x}{3y^2}}{\frac{2x}{9m^3}}$.

5. $\frac{9a^2}{4b} \div 2c^2$.

6. $\frac{x^2-1}{7h} \div \frac{x+1}{2m}$.

7. $\frac{-3x}{4y+1} \div \frac{1}{2x-1}$.

8. $\dfrac{\frac{1}{x}}{\frac{3}{x}}$.

9. $\frac{gt^2}{2} \div \frac{ag^3}{8}$.

10. $pv^3 \div \frac{6p^4}{v^5}$.

11. $\frac{-6}{x^3} \div \frac{-18}{x^7}$.

12. $\dfrac{5a}{\frac{3a^2}{2b}}$.

13. $\dfrac{\dfrac{4x}{7y^2}}{9x^3}$.   14. $\dfrac{\dfrac{1}{t^2}}{t^2}$.   15. $\dfrac{t^2}{\dfrac{1}{t^2}}$.

16. $\left(\dfrac{a^2+b^2}{b}-a\right)\div\left(\dfrac{1}{b}-\dfrac{1}{a}\right)$.

17. $\dfrac{a}{b}+\dfrac{c}{d}\div\left(1-\dfrac{ac}{bd}\right)$.

18. $\left(1-\dfrac{1}{a^3}\right)\div\left(\dfrac{1}{a^2}-\dfrac{1}{a^3}\right)$.

Perform the operations indicated:

19. $x-\dfrac{1}{1+\dfrac{1}{x}}$.   20. $t+\dfrac{1}{\dfrac{1}{t}+1}$.   21. $\left(\dfrac{p-q}{p+q}+1\right)\div\left(1-\dfrac{2q}{p+q}\right)$.

49. Factoring. The process of finding two or more factors whose product is a given algebraic expression, is called **factoring** the given expression.

In division one factor (the divisor) is given and another factor (the quotient) is to be found such that the product of the two factors is the given expression (the dividend). Division is thus really a variety of factoring, although the name "factoring" is not usually applied to it. As a rule, when an expression is to be factored, none of the factors is specified in advance, and any set of factors is acceptable on the sole condition that their product is the given expression.

50. Several type products are of special importance. These have been treated in the FIRST COURSE (pp. 76–95, 175–188), and are collected here for reference.

I. $xy+xz=x(y+z)$.

II. $x^2\pm 2xy+y^2=(x\pm y)^2$.

III. $x^2-y^2=(x+y)(x-y)$.

IV. $x^2+(a+b)x+ab=(x+a)(x+b)$.

V. $x^3+3x^2y+3xy^2+y^3=(x+y)^3$.

VI. $x^3+y^3=(x+y)(x^2-xy+y^2)$.

VII. $x^3-y^3=(x-y)(x^2+xy+y^2)$.

In all of these formulas the letters may, of course, represent either positive or negative numbers.

Detailed treatment of these types is given in the First Course, together with exercises under each type. The following set of miscellaneous exercises covers all the types.

ORAL EXERCISES

Factor:

1. $ax + ay$.
2. $ax + a$.
3. $ax + a^2$.
4. $abx - ay$.
5. $x^2 + ax$.
6. $x^3 + ax^2$.
7. $a^2x^2 - ax$.
8. $mx + my + m$.
9. $px^2 + p^2x + pq$.
10. $x^3 - x^2 + x$.
11. $x^2y^2 + xy + y$.
12. $(a + b)x - (a + b)y$.
13. $s^2t + st^2 + s^2t^2$.
14. $\frac{1}{2}gt^2 + gt$.
15. $abc - acd + bcd$.
16. $a^2x^2 - b^2y^2$.
17. $4x^2 - 9z^2$.
18. $a^2b^2x^2 - c^2$.
19. $1 - p^2q^2y^2$.
20. $m^2p^2 - s^2t^2$.
21. $(a + b)^2 - (c + d)^2$.
22. $1 - (m + p)^2$.
23. $a^2 + 2abc + b^2c^2$.
24. $a^4x^4 + 2a^2x^2 + 1$.
25. $a^2 + 2a(b + c) + (b + c)^2$.
26. $1 - 2(x + y) + (x + y)^2$.
27. $49 + 14x + x^2$.
28. $9 - 12y^2 + 4y^4$.
29. $16 - 40z + 25z^2$.
30. $a^2b^2 + 2abcd + c^2d^2$.
31. $\frac{1}{4} + t + t^2$.
32. $y^4 + .4y^2 + .04$.
33. $m^2x^2 + 4mx + 4$.
34. $\dfrac{z^4}{16} - \dfrac{1}{2}z^2 + 1$.
35. $x^2 - 5x + 6$.
36. $x^2 + 7x + 12$.
37. $x^2 - x - 12$.
38. $x^2 + x - 6$.
39. $t^2 - 12x + 35$.
40. $s^2 - 2x - 35$.
41. $12x^2 + 7x + 1$.
42. $15y^2 - 2x - 1$.
43. $6x^2 - 12x + 6$.
44. $3y^2 + 8y + 5$.
45. $a^3 - b^3$.
46. $a^3 + b^3$.
47. $8 - a^3b^3$.
48. $64 - x^3y^3$.
49. $8x^3 - 12x^2 + 6x - 1$.
50. $27y^3 + 27y^2 + 9y + 1$.

WRITTEN EXERCISES

Factor:

1. $x^2 - 49$.
2. $t^2 - 7t + 6$.
3. $5y^3 - 3y^2 + y$.
4. $4v^2 - 9u^2$.
5. $1 + 10a + 25a^2$.
6. $a^3 + 3a^2b + 3ab^2 + b^3$.
7. $(x+1)^2 + 2(x+1) + 1$.
8. $1 - \frac{2}{a} + \frac{1}{a^2}$.
9. $x^2 - 2x - 15$.
10. $a^2b^2 - 1$.
11. $x^4 + 6x^2y + 9y^2$.
12. $v^8 - t^4$.
13. $1 - x^3y^3$.
14. $a^4 - 4a^2b^2$.
15. $(x+y)^4 - 1$.
16. $a^3b^3 - c^3d^3$.
17. $x^4 - 2x^2y^2 + y^4$.
18. $(m+p)^3 - 1$.
19. $(x+y)^2 - (p+q)^2$.
20. $(a+b)^3 + x^3$.
21. $x^3 + \frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{8}$.
22. $a^3 - 1\frac{1}{2}a^2 + \frac{3}{4}a - \frac{1}{8}$.
23. $x^2 + (a+b)x + ab$.
24. $y^2 + (ac - bd)y - abcd$.
25. $acx^2 + (cb + ad)x + bd$.
26. $m^4p^4 + 4s^4$.
27. $\frac{x^3}{8} - \frac{3x^2}{4} + \frac{3x}{2} - 1$.
28. $.125x^3 + .75x^2 + 1.5x + 1$.
29. $(2m+1)^2 - (3m-1)^2$.
30. $8a^3 + 12a^2t + 6at^2 + t^3$.
31. $g^4 - 16$.
32. $a^2y^2 - 11ay + 24$.
33. $4h^2 - 12ht + 9t^2$.
34. $1 - m^3$.
35. $(2a+b)^2 - (3a-2b)^2$.
36. $8a^3 - 32a^5$.
37. $10m^2x - 60mx + 90x$.
38. $t^3 + 27$.
39. $x^4 + 3x^2 - 28$.
40. $8v^3 - 60v^2w + 150vw^2 - 125w^3$.
41. $s^3 - 64$.
42. $56x^2 - 68x + 20$.
43. $24x^2 + x - 10$.
44. $15x^2 - 34x - 16$.
45. $a^2z^2 - 7az + 12$.
46. $y^4 - 13y^2 + 42$.
47. $(a+1)^2 - (a-1)^2$.
48. $x^2 - \frac{1}{2}x - \frac{1}{2}$.

Calculate:

49. $27^2 - 25^2$.
50. $387^2 - 377^2$.
51. $26^3 - 25^3$.

NOTE. Chapters I and II are themselves in the nature of summaries; consequently, no summaries of these chapters are given here.

REVIEW

WRITTEN EXERCISES

Perform the indicated operations, expressing fractional results in lowest terms:

1. $2a - 3b + 4a + 11c - 2d + 6 - 8a - 9b + 3c - 4 - 5d + 2a - b$.
2. $x^2y + 3xy^2 + 4x^2 - 2xy - 3x^2y + 2x^2y^2 - 4xy + 8xy^2 - 7y^2 - 3x^2y^2$.
3. $5x - 7y - (3x + 4y - 2) + 3 + (8x - 7) - (2x - 8y + 13) + 8(2x - 1)$.
4. $4x - \{2c - (3x + 2y + 5c)\}$.
5. $5ac - \{4b + 2c[3a - 5b - (6a - 2b - \overline{7c + 3})]\}$.

If $L = 4x + 2y - 3$, $M = x - 9y + 1$, $R = y - 5x$, find:

6. $L + M + R$.
7. $3L - 2M + R$.
8. $LM - 3R$.
9. $L^2 - M^2$.
10. $(L + M)^2$.
11. $3L^2 - 5MR$.

If $X = 7a + 2b$, $Y = 2a - 9b + 3$, $Z = 4b - 7a - 3$, find:

12. $X - (3Y - 2Z)$.
13. $2Y - [Z + 3(4X - 8Y)]$.
14. $X^2 - 16Y^2$.
15. $Y^2 + 4YZ$.
16. XYZ.
17. $\dfrac{a+5}{a-5} - \dfrac{a^2+75}{a^2-25} + \dfrac{a-5}{a+5}$.
18. $\dfrac{6m+4p}{15m} - \dfrac{3m^2+4p^2}{15mp} + \dfrac{m-2p}{5p}$.
19. $1 - \dfrac{9+a}{1+a^2} - \dfrac{a-3}{a+1}$.
20. $\left(\dfrac{1}{x} - \dfrac{1}{y}\right)(x+y)$.
21. $\dfrac{2}{t^2+9t+20} + \dfrac{1}{t^2+7t+12} - \dfrac{3}{t^2+11t+28}$.
22. $c^4\left(\dfrac{x^2}{c^3} - \dfrac{3x}{c^2} + \dfrac{14}{c}\right)$.
23. $\dfrac{1}{3a^2}\left(\dfrac{6a^3+9a^2}{4a+2}\right)$.
24. $(b+3)^3 - (b-2)^3$.
25. $(x^n + 3x^{n-1} + 5x^{n-2})(x^2 + 4x)$.
26. $\left(\dfrac{10x^3}{3m^2} - \dfrac{5x^3}{9m^2} - \dfrac{5x^2}{m^4}\right) \div \dfrac{15x}{m^2}$.

27. $\left(\frac{2x^2}{y^2}-\frac{5y^2}{x^2}+3\right)\div\left(\frac{x}{y}-\frac{y}{x}\right).$

28. $\frac{35+12g+g^2}{15+8g+g^2}\div\frac{56+15g+g^2}{24+11g+g^2}.$

29. $\dfrac{\frac{1}{a}-\frac{3}{a^2}}{\frac{2}{a^3}-\frac{3}{a}}.$

30. $\dfrac{\frac{q+1}{q-1}+\frac{q-1}{q+1}}{\frac{q+1}{q-1}-\frac{q-1}{q+1}}.$

31. $\left(\frac{3a}{b}-\frac{11b}{3a}\right)\left(\frac{3a}{b}-\frac{b}{3a}\right)-\left(\frac{3a}{b}-\frac{2b}{3a}\right)^2.$

32. $\frac{2}{x^2+4x+3}+\frac{1}{x^2+3x+2}-\frac{3}{x^2+5x+6}.$

33. $\frac{x^2y}{a^3+1}\div\frac{2x^3}{a+1}.$

34. $\frac{4a+7}{a^2-16x+64}\div\frac{16a^2+56a+49}{a^2-64}.$

35. $\frac{4x}{x-2}+\frac{3}{x^2-4x+4}+\frac{7}{x^2-4}.$

CHAPTER II

EQUATIONS

EQUATIONS WITH ONE UNKNOWN

51. Two algebraic expressions are **equal** when they represent the same number.

52. If two numbers are equal, the numbers are equal which result from:

1. *Adding the same number to each.*
2. *Multiplying each by the same number.*

Subtraction and division are here included as varieties of addition and multiplication.

53. The equality of two expressions is indicated by the symbol, =, called "the sign of equality."

54. Two equal expressions connected by the sign of equality form an **equation**.

55. Such values of the letters as make two expressions equal are said to *satisfy* the equation between these expressions.

56. Equations that are satisfied by any set of values whatsoever for the letters involved are called **identities**.

57. Equations that are satisfied by particular values only are called **conditional** equations, or, when there is no danger of confusion, simply equations.

58. The numbers that satisfy an equation are called the **roots** of the equation.

59. **To solve** an equation is to find its roots.

60. The letters whose values are regarded as unknown are called the **unknowns.**

61. The **degree** of an equation is stated with respect to its unknowns. It is the highest degree to which the unknowns occur in any term in the equation. Unless otherwise stated, all the unknowns are considered.

62. An equation of the first degree is called a **linear** equation.

63. An equation of the second degree is called a **quadratic** equation.

64. An equation of the third or higher degree is called a **higher** equation.

65. In order to state the degree of an equation its terms must be united as much as possible.

66. Terms not involving the unknowns are called **absolute** terms.

67. **Equivalent Equations.** If two equations have the same roots, the equations are said to be equivalent. If two equations have together the same roots as a third equation, the two equations together are said to be equivalent to the third.

68. **The Linear Form, $ax + b$.** Every polynomial of the first degree can be put into the form $ax + b$. That is, by rearranging the terms suitably, it can be written as the product of x by a number not involving x, plus an absolute term. Hence, the form $ax + b$ is called a general form for all polynomials of the first degree in x.

69. Every equation of the first degree in one unknown can be put into the form:

$$ax + b = 0.$$

Consequently this is called *a general equation of the first degree in one unknown.*

70. General Solution. From the equation $ax + b = 0$, (1)

we have $$ax = -b, \tag{2}$$

and hence, $$x = \frac{-b}{a}. \tag{3}$$

71. $-\frac{b}{a}$ is the general form of the root of the equation of the first degree. There is always one root, and only one.

The advantage of a general solution like this is that it leads to a formula which is applicable to all equations of the given form.

In words: *When the equation has been put into the form $ax + b = 0$ the root is the negative of the absolute term divided by the coefficient of x.*

Test. The correctness of a root is tested by substituting it in the *original equation.*

If substituted in any later equation, the work leading to the equation is, of course, not covered by the test.

Results for problems expressed in words should be tested by substitution in the *conditions of the problem.*

If tested by substitution in the equation only, the correctness of the solution is tested, but the setting up of the equation is not tested. Negative results that may occur in such problems are always correct as solutions of the equations, but they are admissible as results in the concrete problem only when the unknown quantity is such that a unit of the unknown quantity is offset by a unit of its opposite.

For example, if the unknown measures distance forward, a negative result means that a corresponding distance backward satisfies the conditions of the problem. But, if the unknown is a number of men, a negative result is inadmissible, since no opposite interpretation is possible.

ORAL EXERCISES

Solve for x:

1. $3x = 15.$	4. $x - 6 = 10$
2. $2x = 11.$	5. $x + 6 = 12.$
3. $4\frac{1}{2}x = 9.$	6. $2x + 1 = 13.$

Solve for t:

7. $6t = 36.$
8. $t - 5 = 20.$
9. $2t + 5 = 25.$
10. $3t - 8 = 22.$
11. $at = ab.$
12. $at + b = c.$

Solve for y:

13. $3y - 1 = 2.$
14. $2y - 1 = 7.$
15. $5y + 5 = 35.$
16. $by = bc.$
17. $by = b + c.$
18. $ay - b = c.$

Solve for a:

19. $6a = 18.$
20. $2\frac{1}{2}a = 10.$
21. $a + 1 = 13.$
22. $3a + 1 = 13.$
23. $2a - 5 = 15.$
24. $5a - b = c.$

WRITTEN EXERCISES

Solve for x:

1. $x - 75 = 136.$
2. $2x - 15 = 45.$
3. $3x + 6 = 48.$
4. $5x - 10 = 55.$
5. $1.5x - 5 = 70.$
6. $3.5x - 5 = 100.$
7. $2.1x - 41 = 400.$
8. $1.3x + .1 = 1.79.$
9. $.25x + .50 = 3.25.$
10. $.11x + .11 = 1.32.$
11. $325x + 60 = 400.$
12. $125x + 5 = 10.$
13. $8x + 625 = 105.$
14. $ax + bx = c.$
15. $abx + ax = ab.$
16. $cx + dx = c + d.$
17. $mx + px = p + q.$
18. $lx + mx = l - m.$
19. $ax + bx = 2(a + b).$
20. $2cx + dx = 1.$

Solve and test:

21. $2(x - 1) + 3(2x + 5) = 0.$

22. $\frac{4(x+2)}{5}+3=2x+1.$

23. $\frac{x-1}{2}+\frac{x-1}{3}+\frac{x-1}{4}=x.$

24. $\frac{3y+1}{4y-2}=\frac{1-3y}{4-4y}.$

25. $\frac{1+3p}{p}=\frac{17-5p}{p}.$

Solve for c:

26. $5c+3a=ac.$

27. $\frac{4c-1}{3m}=\frac{2c+5}{6m}.$

28. $v=ct+\frac{gt^2}{2}.$

29. $(a+c)(a-c)=-(c+a)^2.$

30. $\frac{3c}{c-3}-\frac{c-2}{c-4}=\frac{(c-2)(2c-7)}{c^2-7c+12}.$

31. $\frac{L}{1+ct}=l.$

32. An inheritance of \$2000 is to be divided between two heirs, A and B, so that B receives \$100 less than twice what A receives. How much does each receive?

33. To build a certain staircase 20 steps of a given height are required. If the steps are made 2 inches higher, 16 steps are required. Find the height of the staircase.

34. In a certain hotel the large dining room seats three times as many persons as the small dining room. When the large dining room is $\frac{2}{3}$ full and the small dining room $\frac{1}{2}$ full, there are 100 persons in both together. How many does each room seat?

35. Tickets of admission to a certain lecture are sold at two prices, one 25 cents more than the other. When 100 tickets at the lower price and 60 at the higher price are sold, the total receipts are \$95. Find the two prices.

36. Originally, $\frac{10}{11}$ of the area of Alabama was forest land. One third of this land has been cleared, and now 20 million acres are forest land. Find the area of Alabama in million acres.

37. In a recent year the railroads of the United States owned 70,000 cattle cars. Some of these were single-decked, and others double-decked. There were 44,000 more of the former than of the latter. Find how many there were of each.

38. The average number of sheep carried per deck is 45 larger than the average number of calves. If a double-decked car has the average number of calves on the lower deck and of sheep on the upper deck, it contains 195 animals. Find the number of sheep and of calves.

39. The average number of inhabitants per square mile for Indiana is $\frac{7}{4}$ of that for Iowa, and that for Ohio is 32 greater than that for Indiana, and 62 greater than that for Iowa. Find the number for each state.

40. Lead weighs $\frac{63}{15}$ times as much as an equal volume of aluminium. A certain statuette of aluminium stands on a base of lead. The volume of the base is twice that of the statuette, and the whole weighs 282 oz. Find the weight of the statuette and of the base.

41. A man inherits \$10,000. He invests some of it in bonds bearing $3\frac{1}{2}\%$ interest, the rest in mortgages bearing $5\frac{1}{2}\%$ interest per annum. His entire annual income from these investments is \$510. Find the amount of each investment.

42. A pile of boards consists of inch boards and half-inch boards. There are 80 boards and the pile is 58 in. high. How many boards of each thickness are there?

43. How much water must be added to 30 oz. of a 6% solution of borax to make a 4% solution?

44. How much acid must be added to 10 quarts of a 2% solution to make a 5% solution?

45. A hardware dealer sold a furnace for \$180 at a gain of 5%. What did the furnace cost him.

46. A merchant sold a damaged carpet for \$42.50 at a loss of 15%. What did the carpet cost him?

47. A collector remitted \$475 after deducting a fee of 5%. How many dollars did he collect?

48. The amount of a certain principal at 4% simple interest for 1 year was \$416. What was the principal?

EQUATIONS WITH TWO UNKNOWNS

72. Systems of Equations. Two or more equations considered together are called a **system of equations.**

73. Simultaneous Equations. Two or more equations are said to be **simultaneous** when all of them are satisfied by the same values of the unknowns.

74. All systems of two independent simultaneous equations of the first degree in two unknowns can be solved by the **method of addition and subtraction**, which consists in multiplying one or both of the given equations by such numbers that the coefficients of one of the unknowns become equal. Then by subtraction this unknown is eliminated, and the solution is reduced to that of a single equation.

If the coefficients of one unknown are made numerically equal, but have opposite signs, the equations should be added.

75. Occasionally the **method of substitution** is useful. This consists in expressing one unknown in terms of the other by means of one equation and substituting this value in the other equation, thus eliminating one of the unknowns.

This may be the shorter method when an unknown in either equation has the coefficient 0, $+1$, or -1.

Examples of the use of these methods may be found in the First Course, pp. 117, 119.

General Solution. A general form for two equations of the first degree is

$$ax + by = e, \tag{1}$$

$$cx + dy = f. \tag{2}$$

From these it is possible (without knowing the values of a, b, c, d, e, f) to find a *general form* for the solution, namely:

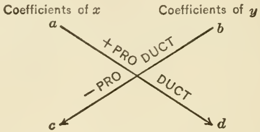
$$x = \frac{de - bf}{ad - bc}, \tag{3}$$

$$y = \frac{af - ce}{ad - bc}. \tag{4}$$

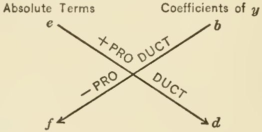
These results are the *formulas* for the roots of any system of two independent linear simultaneous equations with two unknowns.

76. The application of these formulas is made easier by noticing how they are formed from the known numbers in the equations.

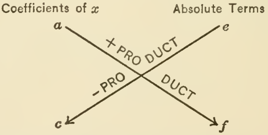
1. The denominator is the same in each result and is made up from the coefficients as follows:



2. The numerator of the value of x is made up thus:



3. The numerator of the value of y:



Examples of the use of these formulas may be found in the First Course, pp. 223–224.

The graphical representation of all the solutions of one equation in two unknowns and the graphical solution of a system of two equations in two unknowns may be reviewed at this point if desired. (See First Course, pp. 170–173, 207–213, 232–234.)

WRITTEN EXERCISES

Solve:

1. $x + y = 5,$
   $x - y = 3.$

2. $x + y = 5,$
   $x - y = 1.$

3. $2x + y = 3,$
   $x + y = 2.$

4. $2x + 3y = 7,$
   $x - y = 1.$

5. $2x + 2y = 8,$
   $2x - y = 2.$

6. $3x - y = -5,$
   $2x - y = -3.$

7. $4x - 3y = 7.$
   $3x - 4y = 7.$

8. $5x + y = 9,$
   $3x + y = 5.$

9. $4x + 5y = 22,$
   $3x + 2y = 13.$

10. $x - 5y = -22,$
    $5x - y = 10.$

11. $4x - 3y = 3,$
    $3x - 4y = -3.$

12. $4x + 2y = 1,$
    $3x - 2y = \frac{5}{2}.$

13. $12x - 11y = 87.$
    $4x + 2y = 46.$

14. $7x - 2y = 3,$
    $7x - 4y = -1.$

15. $9x - 3y = -6,$
    $8x - 2y = -6.$

16. $ax + y = 1,$
    $bx + y = 2.$

17. $ax + by = c,$
    $px + qy = d.$

18. $x - my = a,$
    $x + py = b.$

19. $ax - by = e,$
    $cx - dy = f.$

20. $ax - y = b,$
    $cx + y = d.$

EQUATIONS WITH THREE OR MORE UNKNOWNS

77. The definitions and methods for the solution of two equations with two unknowns may be applied equally well to a greater number of equations and unknowns.

To solve three linear equations with three unknowns, eliminate one unknown from any pair of the equations and the same unknown from any other pair; two equations are thus formed which involve only two unknowns and which may be solved by methods previously given.

Four or more equations with four or more unknowns may be solved similarly.

WRITTEN EXERCISES

Solve and test:

1. $2x + 3y = 19,$
$3x - 4y = 3.$

2. $5x - 2y = 46,$
$x + y = 12.$

3. $8x + 7y = 11,$
$2x - 3y = 17.$

4. $4x - 6y = 1,$
$3x - 5y = 2.$

5. $7x + 9y + 1 = 0,$
$3x + 4y - 4 = 0.$

6. $17x + 13y - 5 = 0,$
$8x + 6y + 3 = 0.$

7. $5x = 3y,$
$2x + 8y = 4.$

8. $ax - by = c,$
$cx + ay = b.$

9. $4a^2x + 5ay = 3,$
$6ax + 7y = 2.$

10. $ax + by = c,$
$a^2x + b^2y = c^2.$

11. $x + by = 1,$
$\frac{x}{a} + y = 3.$

12. $\frac{2}{3}x + \frac{5}{6}y = \frac{1}{2},$
$\frac{3}{8}x + \frac{7}{16}y = \frac{1}{3}.$

13. $2(2x + 3y) - \frac{5(x+3)}{8} - \frac{3y}{4} = 9, \quad x + y = 1.$

14. $4y + 3(y - x - 2) + 20x = 0,$
$3(x - 5) - 2\left(x - \frac{y}{2}\right) + \frac{2y}{3} = 0.$

15. $4x - 2y + z = 3,$
$x + 3y + 2z = 13,$
$-8x + 12y + z = 21.$

16. $5x + 4y + 2z = 17,$
$3x - 2y + 5z = 2,$
$2x - y + 3z = 2.$

17. $x - y - z = a,$
$3y - x - z = 2a,$
$7z - y - x = 4a.$

18. $3x - 2z + 5 = 0,$
$2x + 3y - 21 = 0,$
$4y + 7z - 69 = 0.$

19. $x + y = \frac{1}{a},$
$y + z = \frac{1}{b},$
$x + z = \frac{1}{c}.$

20. $\frac{x}{a+b} + \frac{y}{b-c} + \frac{z}{a+c} = 2c,$
$\frac{x}{a-b} - \frac{y}{b-c} - \frac{z}{a-c} = 2a,$
$\frac{x}{a-b} + \frac{y}{c-b} - \frac{z}{a+c} = 2a - 2c.$

21. $x - y - z - 2w = -12,$
$3x - y - 2z + 8w = 40,$
$4x - 4y + 7z - 5w = 52,$
$3x - y + 2z + w = 44.$

22. A merchant bought a certain number of platters for \$366. Three were broken during shipment. He sold $\frac{1}{6}$ of the remainder at a profit of 25%, for \$75. Find the number of platters bought and the price per platter.

23. A certain hall contains both gas jets and electric lights. When 60 gas jets and 80 electric lights are used, the cost for an evening is \$4. If 90 gas jets and 60 electric lights are used, the cost is \$4.05. Find the cost per gas jet and electric light.

24. A tailor paid \$12 for 4 yd. of cloth and 8 yd. of lining. At another time he paid \$21 for 6 yd. of the cloth and 16 yd. of the lining. Find the price of each per yard.

25. A certain train runs 25 mi. per hour on the level, 15 mi. per hour on up grade, and 30 mi. per hour on down grade. It goes from A to B, 200 mi., in 8 hr. 48 min., and from B to A in 9 hr. 12 min. How many miles are level, up grade, and down grade respectively between A and B?

26. Two wheelmen are 328 ft. apart and ride toward each other. If A starts 3 seconds before B, they meet in 14 seconds after A starts; or if B starts 2 seconds before A, they meet in 14 seconds after B starts. Find the rate of each.

27. A man had a portion of his capital invested in stocks paying 6% dividends, the remainder in mortgages paying 5%. His annual income was \$700. The next year the dividend on the stock was reduced to 5%, but by reinvestment he replaced his old mortgages by new ones paying $5\frac{1}{2}$%. His income for this year was \$690. How much had he invested in stocks; also in mortgages?

28. The sum of the digits in a certain number of two figures is 13, and if the result of multiplying the tens' digit by $1\frac{1}{2}$ is added to the number itself, there results a number with the same digits in reverse order. Find the number.

QUADRATIC EQUATIONS

78. Quadratic Equations. Equations of the second degree are called **quadratic equations.**

A general form for quadratic equations in one unknown is

$$ax^2 + bx + c = 0,$$

in which a, b, c represent any known numbers, except that a may not be zero.

79. Solution of Quadratic Equations. (1) The incomplete quadratic equation $x^2 = a$ is solved by extracting the square root of both members. The roots are: $x = \pm\sqrt{a}$.

(2) The incomplete quadratic equation $ax^2 + bx = 0$ is solved by factoring. The roots are $x = 0$ and $x = -\frac{b}{a}$.

(3) Complete quadratic equations are solved by completing the square.

The process consists of two main parts:

(*a*) *Making the left member a square while the right member does not contain the unknown.*

This is called completing the square.

It is based upon the relation $(x + a)^2 = x^2 + 2\,ax + a^2$, in which it appears that the last term, a^2, is the square of one half of the coefficient of x.

(*b*) *Extracting the square roots of both members and solving the resulting linear equations.*

Square roots which cannot be found exactly should be indicated.

EXAMPLE

Solve:	$x^2 - 8\,x + 9 = 0.$	(1)
Transposing,	$x^2 - 8\,x = -9.$	(2)
Completing the square,	$x^2 - 8\,x + 16 = -9 + 16.$	(3)
Rearranging,	$(x - 4)^2 = 7.$	(4)
Extracting the square root,	$x - 4 = \pm\sqrt{7}.$	(5)
Solving (5) for x,	$x = 4 \pm \sqrt{7}.$	(6)

(4) If any quadratic equation has zero for the right member, and if the polynomial constituting the left member can be factored, the quadratic is equivalent to two linear equations whose roots can readily be found. (See First Course, pp. 186–187.)

ORAL EXERCISES

Solve:

1. $x^2 = 16.$
2. $y^2 = 64.$
3. $z^2 = 8.$
4. $x^2 = a^2.$
5. $y^2 = a^2b^2.$
6. $2x^2 = 8.$
7. $3y^2 = 27.$
8. $5z^2 = 125.$
9. $3t^2 = 75.$
10. $x(x-1) = 0.$
11. $x^2 + x = 0.$
12. $y^2 - 4 = 0.$
13. $x^2 + 2x + 1 = 0.$
14. $y^2 - 2y + 1 = 0.$
15. $z^2 - 3z + 2 = 0.$
16. $3x^2 = 6x.$
17. $(x-1)(x-2) = 0.$
18. $x^2 - 5x + 6 = 0.$
19. $(x-7)(x+1) = 0.$
20. $x^2 + x + \frac{1}{4} = 0.$
21. $s^4 - 16 = 0.$
22. $t^2 - t + \frac{1}{4} = 0.$
23. $\frac{1}{4}p^2 + p + 1 = 0.$
24. $x(x^2 - 2x - 3) = 0.$

WRITTEN EXERCISES

Solve:

1. $3x^2 = 18.$
2. $x^2 - 5x = 0.$
3. $x^2 - 5x + 6 = 0.$
4. $x^2 + 4x - 3 = 0.$
5. $t^2 - 2t - 6 = 0.$
6. $8p^2 = 5p.$
7. $x^2 + 11x + 24 = 0.$
8. $15x^2 - 13x + 5 = 0.$
9. $15y^2 + 134y + 288 = 0.$
10. $\frac{x}{4} = \frac{9}{x-8} + \frac{2x}{5} + 1.35.$
11. $\frac{1}{y-3} + \frac{1}{1-y} - \frac{1}{y-2} = 0.$
12. $7(7-z)(z-6) + 3(5-z)(2-z) - 40 = 0.$
13. $\frac{6-2w}{w-2} + \frac{5+w}{3+w} = \frac{w-5}{2-w}.$

14. Find two numbers whose sum is 10, and the sum of whose squares is 68.

SUGGESTION. Let x represent one number and $10 - x$ the other.

15. A room is 3 yd. longer than it is wide; at \$1.75 per square yard, carpet for the room costs \$49. Find the dimensions of the room.

16. A man bought for \$300 a certain number of oriental rugs at the same price each. If he had bought rugs each costing \$40 more, he would have obtained 2 fewer rugs. How many rugs did he buy?

17. A dealer bought a number of similar tables for \$153. He sold all but 7 of them at an advance of \$1 each on their cost, thus receiving \$100. How many tables did he buy?

18. A man invested \$6000 at a certain rate of simple interest during 4 years. At the end of that time he reinvested the capital and the interest received during the 4 years, at the rate of interest 1% lower than at first. His annual income from the second investment was \$372. What was the original rate of interest?

19. A rectangle of area 84 sq. in. is 5 in. longer than it is wide. Find its dimensions.

20. A certain number of men hire an automobile for \$156. Before they start, two others join them, sharing equally in the expense. The amount to be paid by each of the original hirers is thus reduced by \$13. How many men were there at first?

21. A man rows down a stream a distance of 21 mi. and then rows back. The stream flows at 3 mi. per hour and the man made the round trip in $13\frac{1}{8}$ hours. What was his rate of rowing in still water?

22. The product of a number and the same number increased by 40 is 11,700; what is the number?

23. If each side of a certain square is increased by 5 the area becomes 64; what is the length of a side?

24. Find two numbers whose sum is 16 and the difference of whose squares is 32.

REVIEW

WRITTEN EXERCISES

Solve:

1. $\frac{x+3}{x-1}=\frac{x-1}{x+3}.$

2. $x^2-14x+33=0.$

3. $a+\frac{b}{x}=c.$

4. $2x+3(4x-1)=5(2x+7).$

5. $4-\frac{3}{2x}=6.$

6. $14x-5(2x+4)=3(x+1).$

7. $(5x-2)(6x+1)-(10x+3)(3x+10)=0.$

8. $(2x-3)(x+1)-(3x-7)(x-4)=36.$

9. $\frac{1}{x+2}+\frac{2}{x-2}=\frac{7}{2x-3}.$

10. $2x+5y=6,$
$4x+11y=3.$

11. $5x-3y=1,$
$13x-8y=9.$

12. $\frac{x}{y}=7,$
$2x-10y=3y+2.$

13. $\frac{7y-6}{3x+4}=\frac{1}{2},$
$\frac{4y-5}{2x+1}=\frac{1}{3}.$

14. $\frac{3x+2y+5}{2}+\frac{x-3y-13}{5}+\frac{y-3x+3}{6}=3,$
$\frac{2x-4y+6}{y-5x+11}=-2.$

15. $6z=43-5y,$
$3z=37-4x,$
$4y=55-5x.$

16. $\frac{y-z}{x+y}=\frac{1}{5},$
$\frac{x-z}{y+z}=\frac{2}{3},$
$\frac{x-y}{x+z}=\frac{1}{4}.$

17. $3x+4y+2z=-4,$
$2x-5y-z=9,$
$-4x+2y+3z=-23.$

18. $(z-2)(x+3)=(x-1)(z-1),$
$(z+8)(y-2)-(y+2)(z+2)=0,$
$y(3-2x)+(2y-3)(1+x)=0.$

19. $\frac{3}{2x+10}-\frac{3x^2+14}{7(4-x)}+\frac{1+3x}{5+x}-\frac{3x}{7}=0.$

20. In a certain election 36,785 votes were cast for the three candidates A, B, C. B received 812 votes more than twice as many as A; and C had a majority of one vote over A and B together. How many votes did each receive?

21. In a certain election there were two candidates, A and B. A received 10 votes more than half of all the votes cast. B received 4 votes more than one third of the number received by A. How many votes did each receive?

22. A group of friends went to dine at a certain restaurant. The head waiter found that if he were to place five persons at each table available, four would have no seats, but by placing six at each table, only three persons remained for the last table. How many guests were there, and how many tables were available?

23. A flower bed of uniform width is to be laid out around a rectangular house 20 ft. wide and 36 ft. long. What must be the width of the bed, in order that its area may be one third of that of the ground on which the house stands?

24. Wood's metal, which melts in boiling water, is made up of one half (by weight) of bismuth, a certain amount of lead, half that much zinc, and half as much cadmium as zinc. How much of each in 100 lb. of Wood's metal?

25. If in the preceding exercise three fourths as much cadmium as zinc is used, a metal is formed that melts at a still lower temperature. How many pounds of each constituent metal are there in 100 lb. of this metal?

26. A cask contains 10 gal. of alcohol. A certain number of quarts are drawn out; the cask is then filled up with water and the contents thoroughly mixed. Later twice as many quarts are drawn out as the previous time and the cask filled up with water. There now remains only 4.8 gal. of alcohol in the mixture. How many gallons were drawn out the first time?

27. A certain fraction not in its lowest terms has the value $\frac{1}{2}$. If the numerator is diminished by 2, and the denominator increased by 2, the value of the resulting fraction is the same

as that which results when the numerator of the given fraction is doubled and the denominator multiplied by 5. Find the fraction.

28. A's investments amount to $4000 less than B's, and C's amount to $6000 more than B's. A's average rate of annual income from his investments is one half of 1 % more than B's, and C's is one half of 1 % less than B's, and A's annual income is $80 less than B's, and C's annual income is $120 more than B's. Find the amount of each man's investments, and each man's average rate of income.

CHAPTER III

RADICALS

DEFINITIONS AND PROPERTIES

80. Rational Numbers. Integers and other numbers expressible as the quotient of two integers are called **rational numbers.**

Thus, 5 and .2, which is expressible as $\frac{2}{10}$, are rational numbers.

81. Irrational Numbers. Any number not rational is called an **irrational number.**

Thus, $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{10}$, $\frac{1}{\sqrt{5}}$, $1+\sqrt{3}$, $\sqrt{2}-\sqrt{3}$, π, are irrational numbers.

82. An indicated root of any number is called a **radical.**

Thus, $\sqrt{5}$, $\sqrt[3]{8}$, $\sqrt{\frac{a}{3b}}$, $\sqrt{a+x^2}$, are radicals.

In the present chapter all roots that cannot be exactly extracted by inspection are indicated. Methods for finding approximate numerical values of certain roots are given later.

83. Surd. An irrational number that is an indicated root of a rational number is sometimes called a **surd.**

Thus, $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{7}$, are surds.

84. An expression involving one or more radicals is called a **radical expression.**

Thus, $5+2\sqrt{3}$, $\frac{4}{\sqrt{x}}-1$, $\frac{8+\sqrt{14a}}{2-\sqrt{3b}}$, are radical expressions.

85. Some Properties of Radicals. A few important properties of radicals are given here. The fuller treatment is contained in the chapter on Exponents.

86. I. $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$

For example, $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}.$

That this is true may be seen by squaring both members.

Thus, $(\sqrt{2} \cdot \sqrt{3})(\sqrt{2} \cdot \sqrt{3}) = \sqrt{6} \cdot \sqrt{6},$

or, $\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{3} = \sqrt{6} \cdot \sqrt{6},$

or, $2 \cdot 3 = 6$, which is known to be true.

In the same way, it may be seen that for every a and b, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$

In words:

The product of two square roots is the square root of the product of the numbers.

WRITTEN EXERCISES

Show by squaring that:

1. $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}.$
2. $\sqrt{4} \cdot \sqrt{7} = \sqrt{28}.$
3. $\sqrt{3} \cdot \sqrt{7} = \sqrt{21}.$
4. $\sqrt{5} \cdot \sqrt{11} = \sqrt{55}.$
5. $\sqrt{2a} \cdot \sqrt{3b} = \sqrt{6ab}.$
6. $\sqrt{x^3} \cdot \sqrt{5y} = \sqrt{5x^3y}.$
7. $\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{5} = \sqrt{30}.$
8. $\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c} = \sqrt{abc}.$

87. II. $\sqrt{a^2b} = \sqrt{a^2}\,\sqrt{b} = a\sqrt{b}.$

In words:

Square factors may be taken from under the radical sign.

Thus, $\sqrt{18} = \sqrt{9} \cdot \sqrt{2} = \sqrt{3^2} \cdot \sqrt{2} = 3\sqrt{2}.$

WRITTEN EXERCISES

Take all square factors from under the radical sign:

1. $\sqrt{20}.$
2. $\sqrt{27}.$
3. $\sqrt{50}.$
4. $\sqrt{48}.$
5. $\sqrt{45}.$
6. $\sqrt{75}.$
7. $\sqrt{24}.$
8. $\sqrt{32}.$
9. $\sqrt{12}.$
10. $\sqrt{40}.$
11. $\sqrt{500}.$
12. $\sqrt{128}.$
13. $\sqrt{8a^2}.$
14. $\sqrt{x^3}.$
15. $\sqrt{48x^3y^2}.$
16. $\sqrt{45a^7y^4}.$

88. III. $a\sqrt{b} = \sqrt{a^2} \cdot \sqrt{b} = \sqrt{a^2 b}.$

In words:

Any factor outside the radical sign may be placed under the radical sign provided the factor is squared.

For example:

$$3\sqrt{2} = \sqrt{9} \cdot \sqrt{2} = \sqrt{18}.$$

WRITTEN EXERCISES

Place under one radical sign:

1. $6\sqrt{2}.$	**5.** $3 \cdot \sqrt{7} \cdot 2.$	**9.** $4\sqrt{2} \cdot 3.$	**13.** $t\sqrt{g}.$
2. $5\sqrt{3}.$	**6.** $5 \cdot \sqrt{3} \cdot \sqrt{2}.$	**10.** $b\sqrt{2}.$	**14.** $r\sqrt{\pi r}.$
3. $2 \cdot \sqrt{3}.$	**7.** $2 \cdot \sqrt{3} \cdot \sqrt{11}.$	**11.** $2x\sqrt{3x}.$	**15.** $\frac{x}{3}\sqrt{18xy}.$
4. $5 \cdot \sqrt{7}.$	**8.** $5 \cdot \sqrt{3} \cdot \sqrt{7}.$	**12.** $ab\sqrt{bc}.$	**16.** $a\sqrt{b-a}.$

PROCESSES

89. PREPARATORY.

Read and supply the blanks:

1. $3a + 2a = (\quad)a.$ Similarly, $3\sqrt{2} + 2\sqrt{2} = (\quad)\sqrt{2}.$

2. $5a - 3a = (\quad)a.$ Similarly, $5\sqrt{2} - 3\sqrt{2} = (\quad)\sqrt{2}.$

3. $7\sqrt{3} + 3\sqrt{3} = (\quad)\sqrt{3}.$ **4.** $8\sqrt{5} - 6\sqrt{5} = (\quad)\sqrt{5}.$

5. $\sqrt{75} - \sqrt{12} = 5\sqrt{3} - 2\sqrt{3} = (\quad)\sqrt{3}.$

90. Addition and Subtraction of Radical Expressions. Radicals can be united by addition or subtraction only when the same root is indicated and the expressions under the radical sign are the same in each.

When the expression cannot be put into this form the sum or the difference can only be indicated.

91. *To add or subtract radical expressions having the same radical part, add or subtract the coefficients of their radical parts.*

For example:

$$2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}.$$

$$2\sqrt{12} + \sqrt{300} = 4\sqrt{3} + 10\sqrt{3} = 14\sqrt{3}.$$

Add $\sqrt{2}$, $-\sqrt{8}$, $\sqrt[3]{16}$, $\sqrt[3]{-54}$:

$$-\sqrt{8} = -2\sqrt{2};\quad \sqrt[3]{16} = 2\sqrt[3]{2};\quad \sqrt[3]{-54} = -3\sqrt[3]{2}.$$

$$\therefore \sqrt{2} - \sqrt{8} + \sqrt[3]{16} + \sqrt[3]{-54} = \sqrt{2} - 2\sqrt{2} + 2\sqrt[3]{2} - 3\sqrt[3]{2}$$
$$= -\sqrt{2} - \sqrt[3]{2}.$$

If the numerical value of the sum or the difference is needed, it can be found approximately by methods given later.

WRITTEN EXERCISES

Find the sum:

1. $\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$.
2. $\sqrt{75}$, $-\sqrt{12}$, $-\sqrt{3}$.
3. $\sqrt{8}$, $\sqrt{5}$, $-\sqrt{18}$.
4. $\sqrt{128}$, $-2\sqrt{50}$, $\sqrt{72}$.
5. $\sqrt[3]{40}$, $-\sqrt[3]{320}$, $\sqrt[3]{135}$.
6. $8\sqrt{48}$, $-\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$.
7. $\sqrt[3]{72}$, $-3\sqrt[3]{9}$, $6\sqrt[3]{243}$.
8. $\sqrt{6}$, $\sqrt{24}$, $\sqrt{63}$.
9. $\sqrt{108}$, $-\sqrt{12}$, $\sqrt{48}$.
10. $\sqrt{75}$, $\sqrt{48}$, $-\sqrt{27}$.
11. $\sqrt{80}$, $\sqrt{20}$, $-\sqrt{45}$.
12. $\sqrt{44}$, $-\sqrt{99}$, $\sqrt{121}$.
13. $5\sqrt{24}$, $-\sqrt{54}$, $3\sqrt{96}$.
14. $\sqrt[3]{27x^4}$, $-\sqrt[3]{64x^4}$, $\sqrt[3]{16x^4}$.

92. Multiplication of Radical Expressions containing Square Roots. In multiplying expressions containing indicated square roots, make use of the relation $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$.

EXAMPLES

1.
$$\begin{array}{l} 3 - 4\sqrt{5} \\ \underline{6 + 2\sqrt{3}\qquad\qquad} \\ 18 - 24\sqrt{5} \\ \underline{\qquad\qquad 6\sqrt{3} - 8\sqrt{15}.} \\ 18 - 24\sqrt{5} + 6\sqrt{3} - 8\sqrt{15}. \end{array}$$

2.
$$\begin{array}{l} 2 - \sqrt{3} \\ \underline{5 + 2\sqrt{3}} \\ 10 - 5\sqrt{3} \\ \underline{\qquad 4\sqrt{3} - 6} \\ 10 - \sqrt{3} - 6 = 4 - \sqrt{3}. \end{array}$$

WRITTEN EXERCISES

Multiply:

1. $2+\sqrt{5}$ by $2-\sqrt{5}$.
2. $1+\sqrt{3}$ by $2+\sqrt{5}$.
3. $2+\sqrt{3}$ by $2+\sqrt{3}$.
4. $\sqrt{2}+\sqrt{3}$ by $1-\sqrt{3}$.
5. $\sqrt{3}-\sqrt{5}$ by $\sqrt{3}+\sqrt{5}$.
6. $\sqrt{5}-\sqrt{6}$ by $\sqrt{5}-\sqrt{6}$.
7. $4+\sqrt{5}$ by $\sqrt{10}$.
8. $3-\sqrt{15}$ by $2+\sqrt{5}$.
9. $1+\sqrt{2}$ by $1-\sqrt{8}$.
10. $2\sqrt{3}-3\sqrt{5}$ by $\sqrt{3}-\sqrt{5}$.
11. $\sqrt{14}+\sqrt{7}$ by $\sqrt{8}-\sqrt{21}$.
12. $\sqrt{5}-\sqrt{48}$ by $\sqrt{5}+\sqrt{12}$.

93. Division of Square Roots. The quotient of the square roots of two numbers is the square root of the quotient of the numbers. In symbols, $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$.

Thus, $\dfrac{\sqrt{5}}{\sqrt{6}}=\sqrt{\dfrac{5}{6}}$, because, multiplying each member by itself,

$$\frac{\sqrt{5}}{\sqrt{6}}\cdot\frac{\sqrt{5}}{\sqrt{6}}=\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{5}{6}},$$

or $$\frac{\sqrt{5}\cdot\sqrt{5}}{\sqrt{6}\cdot\sqrt{6}}=\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{5}{6}},$$

or $$\frac{\sqrt{5^2}}{\sqrt{6^2}}=\sqrt{\left(\frac{5}{6}\right)^2},$$

or $$\frac{5}{6}=\frac{5}{6},$$ which is evidently true.

ORAL EXERCISES

Read each of the following as a fraction under one radical sign:

1. $\dfrac{\sqrt{3}}{\sqrt{5}}$.
2. $\dfrac{\sqrt{3}}{\sqrt{7}}$.
3. $\dfrac{\sqrt{5}}{\sqrt{7}}$.
4. $\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{1}}{\sqrt{2}}$.
5. $\dfrac{1}{\sqrt{5}}$.
6. $\dfrac{\sqrt{2}}{\sqrt{8}}$.
7. $\dfrac{\sqrt{5}}{\sqrt{15}}$.
8. $\dfrac{\sqrt{10}}{\sqrt{20}}$.

94. Rationalizing the Denominator. Multiplying both numerator and denominator of a fraction by an expression that will make the denominator rational is called **rationalizing the denominator.**

Thus, multiplying both numerator and denominator of $\frac{\sqrt{3}}{\sqrt{2}}$ by $\sqrt{2}$, we obtain

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{6}}{2}.$$

WRITTEN EXERCISES

Rationalize the denominator of:

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{1}{\sqrt{3}}$.
3. $\frac{1}{\sqrt{5}}$.
4. $\frac{2}{\sqrt{3}}$.
5. $\frac{3}{\sqrt{7}}$.
6. $\frac{\sqrt{2}}{\sqrt{3}}$.
7. $\frac{\sqrt{5}}{\sqrt{7}}$.
8. $\frac{5}{\sqrt{3}}$.
9. $\frac{\sqrt{5}}{\sqrt{3}}$.
10. $\frac{10}{\sqrt{5}}$.
11. $\frac{6}{\sqrt{3}}$.
12. $\frac{8}{\sqrt{2}}$.

95. Rationalizing Factors. When the denominator is of the form $\sqrt{a}+\sqrt{b}$ or $a+\sqrt{b}$, the rationalizing factor is the same binomial with the connecting sign changed, often called the **conjugate binomial.**

It is not necessary in elementary algebra to take up the rationalizing of more complicated denominators.

EXAMPLES

1. Rationalize the denominator in $\frac{3}{2-\sqrt{5}}$.

The conjugate of $2-\sqrt{5}$ is $2+\sqrt{5}$.

Then, $\frac{3}{2-\sqrt{5}} = \frac{3(2+\sqrt{5})}{(2-\sqrt{5})(2+\sqrt{5})} = \frac{6+3\sqrt{5}}{4-5} = -(6+3\sqrt{5})$.

2. Rationalize the denominator in $\dfrac{2+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$.

The conjugate of $\sqrt{3}+\sqrt{5}$ is $\sqrt{3}-\sqrt{5}$.

Then, $$\frac{2+\sqrt{3}}{\sqrt{3}+\sqrt{5}}=\frac{(2+\sqrt{3})(\sqrt{3}-\sqrt{5})}{(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5})}=\frac{2\sqrt{3}+3-2\sqrt{5}-\sqrt{15}}{3-5}$$

$$=-\frac{3+2\sqrt{3}-2\sqrt{5}-\sqrt{15}}{2}.$$

WRITTEN EXERCISES

Rationalize the denominators:

1. $\dfrac{2+\sqrt{3}}{3+\sqrt{3}}$.
2. $\dfrac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
3. $\dfrac{1}{1-\sqrt{2}}$.
4. $\dfrac{3}{2-\sqrt{5}}$.
5. $\dfrac{5}{\sqrt{3}+\sqrt{7}}$.
6. $\dfrac{2-\sqrt{5}}{3-\sqrt{5}}$.
7. $\dfrac{3}{\sqrt{5}+\sqrt{2}}$.
8. $\dfrac{\sqrt{5}+2\sqrt{2}}{4-2\sqrt{2}}$.
9. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.
10. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.
11. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}-1}$.
12. $\dfrac{2\sqrt{15}-6}{\sqrt{5}+2\sqrt{2}}$.

RADICAL EQUATIONS

96. *To solve equations in which only a single square root occurs, transpose so that the square root constitutes one member. Square both members and solve the resulting equation.*

EXAMPLE

Solve: $$2x-3=\sqrt{x^2+6x-6}. \qquad (1)$$

Squaring both members, $$4x^2-12x+9=x^2+6x-6. \qquad (2)$$

Collecting terms, $$3x^2-18x+15=0. \qquad (3)$$

Dividing both members by 3, $$x^2-6x+5=0. \qquad (4)$$

Solving (4), $$x=5 \text{ or } 1. \qquad (5)$$

TEST. On trial, it appears that 5 satisfies the given equation, taking the radical as positive, while 1 satisfies the equation $2x-3=-\sqrt{x^2+6x-6}$.

1. It must be remembered that the equation resulting from squaring will usually not be *equivalent* to the given equation (FIRST COURSE, Sec. 232, p. 186). It may have additional roots, and trial must determine which of the roots found satisfy the given equation.

2. In order that the given problem may be definite, the radical must be taken with a given sign. If every possible square root is meant, two different equations are really given for solution. Thus, unless restricted, $2x = \sqrt{4-6x}$ is a compact way of uniting the two different equations, $2x = +\sqrt{4-6x}$, and $2x = -\sqrt{4-6x}$. If solved as indicated above, it appears that the first is satisfied when $x = \frac{1}{2}$, the second when $x = -2$.

3. In the exercises of the following set the radical sign is to be understood to mean the *positive* square root.

WRITTEN EXERCISES

Solve:

1. $x = \sqrt{10x+7}$.
2. $x = \sqrt{b+x-bx}$.
3. $3x - 7\sqrt{x} = -2$.
4. $\sqrt{x+5} - x = -1$.
5. $\dfrac{\sqrt{x}-1}{3} = \dfrac{x}{16}$.
6. $x + 5\sqrt{37-x} = 43$.
7. $30 = x - 29\sqrt{x}$.
8. $x = 2 + \sqrt{3-11x}$.
9. $x - \sqrt{x+2} = 3$.
10. $8x + 1 = \sqrt{x+3}$.
11. $\sqrt{100-x^2} = 10 - x$.
12. $x + \sqrt{2s - x^2} = 6$.

SUMMARY

I. **Definitions.**

1. *Rational numbers* are integers and other numbers expressible as the quotient of two integers. Sec. 80.

2. An *irrational number* is any number not rational. Sec. 81.

3. A *radical* is an indicated root of a number. Sec. 82.

4. A *surd* is an irrational number that is an indicated root of a rational number. Sec. 83.

5. A *radical expression* is an expression involving one or more radicals. Sec. 84.

II. **Properties and Operations.**

1. The product of two square roots is the square root of the product of the numbers. Sec. 86.

2. Square factors may be taken from under the radical sign. Sec. 87.

3. Any factor outside the radical sign may be placed under the radical sign provided the factor is squared. Sec. 88.

4. Radical expressions whose radical parts are the same may be added or subtracted by adding or subtracting the coefficients of their radical parts. Sec. 91.

5. The quotient of two square roots is the square root of the quotient of the numbers. Sec. 93.

6. Multiplying both numerator and denominator of a fraction by a factor that will make the denominator rational is called *rationalizing the denominator*. Sec. 94.

7. Radical equations involving only a single square root are solved by transposing so that the square root constitutes one member, squaring and solving the resulting equation. Sec. 96.

REVIEW

WRITTEN EXERCISES

Simplify:

1. $\dfrac{\sqrt{5}}{\sqrt{60}}$.

2. $\sqrt{\frac{27}{8}} - \sqrt{\frac{3}{8}}$.

3. $\sqrt{6} \cdot \sqrt{125}$.

4. $\dfrac{\sqrt{10}}{\sqrt{40}}$.

5. $\sqrt{50} + \sqrt{128}$.

6. $\sqrt{3} \div \sqrt{5}$.

7. $\dfrac{3\sqrt{3} + 2\sqrt{2}}{\sqrt{3} - \sqrt{2}}$.

8. $\sqrt{6} \div \sqrt{2}$.

9. $(2 + \sqrt{3})^2$.

10. $(5 + \sqrt{7})(5 - \sqrt{7})$.

11. $(2\sqrt{3} + 3\sqrt{5}) \div \sqrt{15}$.

12. $(\sqrt{6} + \sqrt{15})(\sqrt{8} - \sqrt{20})$.

Solve:

13. $\sqrt{x+5} = x - 7$.

14. $\sqrt{2x+7} = \dfrac{x}{3}$.

15. $\dfrac{x-10}{2} + \dfrac{x-1}{3} + \dfrac{\sqrt{2x-1}}{2} = 0$.

16. $x + \sqrt{x+5} = 2x - 1$.

17. $x - 2 + \sqrt{2-x} = 0$.

18. $x + \sqrt{9-x^2} = 4$.

SUPPLEMENTARY WORK

ADDITIONAL EXERCISES

Express with rational denominators, and with at most one radical sign in the dividend:

1. $\sqrt{12} \div \sqrt{3}$.
2. $\sqrt{7} \div \sqrt{11}$.
3. $2\sqrt{24} \div 2\sqrt{6}$.
4. $2 \div 3\sqrt{5}$.
5. $\dfrac{4}{\sqrt{5}-1}$.
6. $1 \div (\sqrt{2}-10)$.
7. $\sqrt{2} \div (\sqrt{2}-\sqrt{3})$.
8. $(2\sqrt{6}+5\sqrt{12}) \div \sqrt{6}$.
9. $(5\sqrt{18}-8\sqrt{50}) \div 2\sqrt{2}$.
10. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.
11. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.
12. $\dfrac{\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.
13. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}-1}$.
14. $\dfrac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}$.

Square Root of Binomials of the Form $a+\sqrt{b}$

Binomials of the form $a+\sqrt{b}$ can often be put into the form $x+y+2\sqrt{xy}$, or $(\sqrt{x}+\sqrt{y})^2$, and hence the square root, $\sqrt{x}+\sqrt{y}$, of the binomial can be written at once.

EXAMPLES

1. Find the square root of $4+2\sqrt{3}$.

$$4+2\sqrt{3}=3+1+2\sqrt{3\cdot 1}.$$

Hence, $x+y=3+1$ and $xy=3\cdot 1$, from which $x=3$ and $y=1$.

$$\therefore \sqrt{4+2\sqrt{3}}=\pm(\sqrt{3}+\sqrt{1})=\pm(\sqrt{3}+1).$$

The coefficient of the radical must be made 2 in order to apply the formula $x+y+2\sqrt{xy}$.

2. Find the square root of $3 - \sqrt{8}$.

$$3 - \sqrt{8} = 3 - \sqrt{4 \cdot 2} = 3 - 2\sqrt{2}.$$

$\therefore x + y = 3$, and $xy = 2$. $\therefore x = 2$, $y = 1$ by inspection.

$$\therefore \sqrt{3 - \sqrt{8}} = \pm(\sqrt{2} - \sqrt{1}) = \pm(\sqrt{2} - 1).$$

3. Find the square root of $7 + 4\sqrt{3}$.

$7 + 4\sqrt{3} = 7 + 2\sqrt{4 \cdot 3}$; $x + y = 7$, $xy = 12$; $\therefore x = 4$, $y = 3$.

$$\therefore \sqrt{7 + 4\sqrt{3}} = \pm(\sqrt{4} + \sqrt{3}) = \pm(2 + \sqrt{3}).$$

The square root as a whole may be taken positively or negatively, as in the case of rational roots.

The solution of these problems depends upon finding two numbers whose sum and product are given. This can sometimes be done by inspection, but the general problem is one of simultaneous equations. See Sec. 168, p. 132.

WRITTEN EXERCISES

Find the square root of:

1. $11 + 6\sqrt{2}$.
2. $8 - 2\sqrt{15}$.
3. $49 - 12\sqrt{10}$.
4. $41 - 24\sqrt{2}$.
5. $2\frac{1}{4} - \sqrt{5}$.
6. $2\frac{1}{3} - \frac{4}{3}\sqrt{3}$.
7. $17 + 12\sqrt{2}$.
8. $\frac{3}{2}\sqrt{5} + 3\frac{1}{2}$.
9. $56 - 24\sqrt{5}$.

CHAPTER IV

EXPONENTS

LAWS OF EXPONENTS

97. PREPARATORY.

1. What is the meaning of a^2? Of a^4? Of a^7? Of a^{11}? Of a^n?

2. What is the meaning of $\sqrt{a^2}$? Of $\sqrt[3]{a^3}$? Of $\sqrt[10]{a^{10}}$? Of $\sqrt[n]{a^n}$?

3. $a^2 \cdot a^3 = ?$ $a^3 \cdot a^2 = ?$ $a^5 \cdot a^3 = ?$ $a^8 \cdot a^5 = ?$

4. $a^3 \div a^2 = ?$ $a^4 \div a^2 = ?$ $a^5 \div a^2 = ?$ $b^{10} \div b^4 = ?$

5. $(a^2)^2 = ?$ $(a^2)^3 = ?$ $(c^5)^2 = ?$ $(x^{10})^3 = ?$

98. We shall soon define negative and fractional exponents, but until this is done literal exponents are to be understood to represent positive integers.

99. Law of Exponents in Multiplication.

I. $$a^m \cdot a^r = a^{m+r}.$$

For $$a^m = a \cdot a \cdot a \cdots \text{ to } m \text{ factors},$$

and $$a^r = a \cdot a \cdot a \cdots \text{ to } r \text{ factors}.$$

$$\begin{aligned} \therefore a^m \cdot a^r &= (a \cdot a \cdot a \cdots \text{ to } m \text{ factors})(a \cdot a \cdot a \cdots \text{ to } r \text{ factors}) \\ &= a \cdot a \cdot a \cdot a \cdots \text{ to } m + r \text{ factors} \\ &= a^{m+r}, \text{ by the definition of exponent.} \end{aligned}$$

Similarly, $$a^m \cdot a^r \cdot a^p \cdots = a^{m+r+p} \cdots.$$

ORAL EXERCISES

Multiply:

1. $a^2 \cdot a^4$.
2. $a^3 \cdot a^3$.
3. $a^5 \cdot a^7$.
4. $m^1 \cdot m^5$.
5. $x \cdot x$.
6. $2^3 \cdot 2^4$.
7. $(-1)^3 \cdot (-1)^5$.
8. $6^2 \cdot 6^2$.
9. $5 \cdot 5 \cdot 5^2$.
10. $2^3 \cdot 2^3 \cdot 2^2$.
11. $7 \cdot 7^2 \cdot 7^3$.
12. $3 \cdot 3^5 \cdot 3^2$.
13. $(-1)^2 \cdot (-1)^3 \cdot (-1)^5$.
14. $(-a)^2 \cdot (-a)^4 \cdot (-a)$.

100. Law of Exponents in Division.

II. $$\frac{a^m}{a^r} = a^{m-r}, \text{ if } m > r.$$

For $a^m = a \cdot a \cdot a \cdots m$ factors,

and $a^r = a \cdot a \cdot a \cdots r$ factors.

$$\therefore \frac{a^m}{a^r} = \frac{a \cdot a \cdots m \text{ factors}}{a \cdot a \cdots r \text{ factors}}$$

$= a \cdot a \cdot a \cdots m - r$ factors, canceling the r factors from both terms

$= a^{m-r}$, by definition of exponent.

ORAL EXERCISES

Divide:

1. $\frac{a^4}{a^2}$.
2. $\frac{a^5}{a^3}$.
3. $\frac{a^7}{a^5}$.
4. $\frac{2^4}{2^2}$.
5. $\frac{a^8}{a^5}$.
6. $\frac{(-a)^5}{(-a)}$.
7. $\frac{(-1)^5}{(-1)^3}$.
8. $\frac{(ab)^2}{(ab)}$.
9. $\frac{6^3}{6^2}$.
10. $\frac{5^4}{5^3}$.
11. $\frac{4\pi r^2}{4r}$.
12. $\frac{x^5}{x}$.
13. $\frac{x^2y}{x}$.
14. $\frac{mv^2}{v}$.
15. $\frac{\frac{1}{2}gt^2}{t}$.
16. $\frac{\pi r^3}{r}$.

101. Laws of Exponents for Powers.

III. $$(a^m)^r = a^{mr}.$$

For $(a^m)^r = a^m \cdot a^m \cdot a^m \cdots$ to r factors

$= (a \cdot a \cdots$ to m factors$)(a \cdot a \cdots$ to m factors$)$ to r such parentheses

$= a \cdot a \cdot a \cdots$ to mr factors

$= a^{mr}$, by definition of exponent.

ORAL EXERCISES

Apply this law to:

1. $(4^3)^2$.
2. $(3^2)^5$.
3. $(2^5)^4$.
4. $(a^3)^2$.
5. $(a^2)^3$.
6. $(a^5)^2$.
7. $(x^2)^5$.
8. $(x^3)^3$.
9. $(y^4)^4$.
10. $[(b)^4]^5$.
11. $[(-a)^5]^2$.
12. $[(-8)^2]^3$.

IV. $$(ab)^n = a^n b^n.$$

For $(ab)^n = (ab)\cdot(ab) \cdots$ to n factors

$= (a \cdot a \cdots$ to n factors$)(b \cdot b \cdots$ to n factors$)$

$= a^n b^n$, by definition of exponent.

Similarly, $(abc \cdots)^n = a^n b^n c^n \cdots$.

ORAL EXERCISES

Apply this law to:

1. $(8 \cdot 3)^2$.
2. $(4 \cdot 5)^2$.
3. $(2 \cdot 5)^3$.
4. $(ab)^5$.
5. $(cd)^3$.
6. $(abc)^4$.
7. $(mn)^p$.
8. $(xy)^a$.
9. $(ab)^{2r}$.
10. $(x^2y)^3$.
11. $(x^a y^2)^5$.
12. $(ax^3)^b$.

V. $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

For $\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \cdot \cdot$ to n factors

$= \dfrac{a \cdot a \cdots \text{ to } n \text{ factors}}{b \cdot b \cdots \text{ to } n \text{ factors}}$

$= \dfrac{a^n}{b^n}$, by definition of exponent.

ORAL EXERCISES

Apply this law to:

1. $(\frac{2}{3})^2$.
2. $(\frac{1}{5})^3$.
3. $(\frac{2}{5})^5$.
4. $(\frac{3}{5})^4$.
5. $(-\frac{6}{7})^2$.
6. $(-\frac{3}{4})^3$.
7. $\left(\frac{a}{b}\right)^2$.
8. $\left(\frac{a}{b}\right)^5$.
9. $\left(\frac{c}{d}\right)^3$.
10. $\left(\frac{x}{y}\right)^n$.
11. $\left(\frac{m}{n}\right)^a$.
12. $\left(\frac{p}{q}\right)^x$.
13. $\left(\frac{2a}{b}\right)^5$.
14. $\left(\frac{c}{5d}\right)^7$.
15. $\left(\frac{-xy}{z}\right)^5$.

102. Collected Laws of Exponents.

I.	$a^m \cdot a^r = a^{m+r}$.		Sec. 99.
II.	$a^m \div a^r = a^{m-r}$.	$(m > r.)$	Sec. 100.
III.	$(a^m)^r = a^{mr}$.		Sec. 101.
IV.	$(ab)^n = a^n b^n$.		Sec. 101.
V.	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.		Sec. 101.

FRACTIONAL EXPONENTS

103. Hitherto we have spoken only of positive integers as exponents, the exponent meaning the number of times the base is used as a factor. This meaning does not apply to fractional and negative exponents, because it does not mean anything to speak of using a as a factor $\frac{2}{3}$ of a time, or -6 times. But it is possible to find meanings for fractional and negative exponents such that they will conform to the laws of integral exponents.

104. Preparatory.

Find the meaning of $a^{\frac{1}{2}}$.

Assuming that Law I applies, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a,$

or, $(a^{\frac{1}{2}})^2 = a.$

That is, $a^{\frac{1}{2}}$ is one of the two equal factors of a,

or, $a^{\frac{1}{2}} = \sqrt{a}.$

Thus, the fractional exponent $\frac{1}{2}$ means *square root.*

Similarly, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a,$

or, $(a^{\frac{1}{3}})^3 = a.$

That is, $a^{\frac{1}{3}}$ is one of the three equal factors of a,

or, $a^{\frac{1}{3}} = \sqrt[3]{a}.$

Thus, the fractional exponent $\frac{1}{3}$ means *cube root.*

WRITTEN EXERCISES

Find similarly the meaning of:

1. $a^{\frac{1}{4}}$.
2. $a^{\frac{1}{5}}$.
3. $x^{\frac{1}{8}}$.
4. $b^{\frac{1}{10}}$.
5. $n^{\frac{1}{8}}$.
6. $c^{\frac{1}{7}}$.
7. $m^{\frac{1}{6}}$.
8. $a^{\frac{1}{50}}$.

105. The meaning of $a^{\frac{1}{n}}$ is found as follows:

Assuming that Law I applies,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots \text{to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\cdots \text{ to } n \text{ terms.}}$$
$$= a^{n \cdot \frac{1}{n}}$$
$$= a^1 = a.$$

That is, $a^{\frac{1}{n}}$ is one of n equal factors of a, or $a^{\frac{1}{n}} = \sqrt[n]{a}$.

106. Preparatory.

Find the meaning of $a^{\frac{2}{3}}$.

Assuming that Law I applies, $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$, or $(a^{\frac{2}{3}})^3 = a^2$.
That is, $a^{\frac{2}{3}}$ is one of the three equal factors of a^2, or $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

107. The meaning of $a^{\frac{p}{q}}$ is found as follows:

Assuming that Law I applies,

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots \text{to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}\cdots \text{ to } q \text{ terms}}$$
$$= a^{q \cdot \frac{p}{q}} = a^p.$$

That is, $a^{\frac{p}{q}}$ is one of the q equal factors of a^p,

or, $$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Similarly, $$a^{\frac{p}{q}} = a^{\frac{1}{q} \cdot p} = (\sqrt[q]{a})^p.$$

In words:

a *with the exponent* $\frac{\text{p}}{\text{q}}$ *denotes the* q*th root of the* p*th power of* a, *or the* p*th power of the* q*th root of* a.

This definition applies when p and q are positive integers. The meaning of negative fractional exponents is found in Sec. 116, p. 64.

ORAL EXERCISES

1. State the meaning of:

1. $5^{\frac{2}{3}}$. 3. $6^{\frac{1}{2}}$. 5. $4^{\frac{5}{7}}$. 7. $8^{\frac{3}{4}}$.

2. $a^{\frac{3}{4}}$. 4. $a^{\frac{2}{5}}$. 6. $b^{\frac{5}{6}}$. 8. $c^{\frac{4}{3}}$.

Find the value of:

9. $8^{\frac{4}{3}}$. 10. $16^{\frac{3}{4}}$. 11. $25^{\frac{3}{2}}$. 12. $32^{\frac{3}{5}}$.

WRITTEN EXERCISES

Express with fractional exponents:

1. $\sqrt[4]{a^3}$.
2. $\sqrt[5]{a^2}$.
3. $\sqrt[3]{mn}$.
4. $\sqrt{a}\sqrt[3]{b}$.
5. $\sqrt[3]{ab}$.
6. $\sqrt[3]{ab^2xy}$.
7. $\sqrt[3]{\dfrac{x^4y^2}{4}}$.
8. $\sqrt{\dfrac{9\,a^2b}{x}}$.
9. $\sqrt[8]{16\,x^4y^8}$.
10. $\sqrt{m}\sqrt[3]{n^2}\sqrt[5]{p}$.
11. $\sqrt[5]{\dfrac{32\,a^5b^8}{c^{15}}}$.
12. $\sqrt[3]{\dfrac{16\,a^2}{9\,b^2}}$.
13. $\sqrt[3]{\dfrac{-8\,x^3y^6}{5}}$.
14. $\sqrt[6]{\dfrac{64\,x^{12}}{y}}$.
15. $\sqrt{a+b}$.
16. $\sqrt{a^3+b^3}$.
17. $\sqrt[3]{b^2}\cdot\sqrt{a}$.
18. $\sqrt[5]{-a}\cdot\sqrt[5]{-b}$.
19. $\sqrt[n]{a}$.
20. $\sqrt[3]{(a-b)^2}$.
21. $\sqrt[n]{a}\cdot\sqrt[m]{b}$.
22. $\sqrt[n]{a^m}$.
23. $\sqrt[m]{a^n}$.
24. $\sqrt[pn]{b^{2n}}$.
25. $\sqrt[2m]{a^{mn}}$.
26. $\sqrt[2mn]{a^{2mn}}$.
27. $\sqrt[mn]{a^nb^m}$.
28. $\sqrt[n]{a}\cdot\sqrt[p]{q}$.
29. $\sqrt[2n]{abc}$.
30. $\sqrt[p]{a^q}\cdot\sqrt[q]{a^p}$.

108. The definition of positive fractional exponents has been found as a consequence of the assumption that Law I applies to them. It can be shown that the other laws of Sec. 102, p. 56, also apply to this class of exponents, as thus defined, and we shall so apply them, although the proof is omitted here.

109. According to Law I (Sec. 99), *when the bases are the same, the exponent of the product is found by adding the exponents.*

For example: $a^{\frac{1}{2}} \cdot a^{\frac{3}{4}} = a^{\frac{1}{2}+\frac{3}{4}} = a^{\frac{5}{4}}.$

A general formula for this statement is,

$$a^{\frac{m}{n}} \cdot a^{\frac{p}{q}} = a^{\frac{mq}{nq}+\frac{np}{nq}} = a^{\frac{mq+np}{nq}}.$$

The number a, or the base, must be the same in all factors. When it is not, as in $a^{\frac{1}{2}} \cdot b^{\frac{3}{4}}$, the product cannot be found by adding the exponents.

WRITTEN EXERCISES

Find the products:

1. $a^{\frac{1}{2}} \cdot a^{\frac{2}{3}}.$
2. $4^2 \cdot 4^{\frac{3}{5}}.$
3. $7^{\frac{1}{2}} \cdot 7^{\frac{3}{5}}.$
4. $a^{\frac{3}{2}} \cdot a^{\frac{2}{3}}.$
5. $b^{\frac{1}{2}} \cdot b^{\frac{1}{8}}.$
6. $a^{\frac{2}{3}} \cdot a^{\frac{3}{4}}.$
7. $m^{\frac{3}{4}} \cdot m^{\frac{3}{5}}.$
8. $a^{\frac{5}{3}} \cdot a^{\frac{1}{2}}.$
9. $r^{\frac{1}{n}} \cdot r^{\frac{1}{n}}.$
10. $a^{\frac{1}{n}} \cdot a^{\frac{1}{m}}.$
11. $x^{\frac{1}{a}} \cdot x^{\frac{1}{b}}.$
12. $x^{\frac{m}{n}} \cdot x^{\frac{p}{n}}.$
13. $p^{\frac{2}{3}} \cdot p^{\frac{1}{2}} \cdot p.$
14. $a^{\frac{4}{5}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{1}{2}}.$

110. According to Law II (Sec. 102), *when the bases are the same, the exponent of the quotient is found by taking the difference between the exponents.*

For example: $a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{2}{3}-\frac{1}{2}} = a^{\frac{1}{6}}.$

$$a \div a^{\frac{3}{4}} = a^{1-\frac{3}{4}} = a^{\frac{1}{4}}.$$

A general formula for this statement is,

$$a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}} = a^{\frac{mq-np}{nq}}.$$

$\frac{m}{n}$ is here supposed to be greater than $\frac{p}{q}$, but this restriction will be removed later.

WRITTEN EXERCISES

Find the quotients:

1. $a^{\frac{3}{4}} \div a^{\frac{1}{2}}$.
2. $a \div a^{\frac{7}{12}}$.
3. $a \div a^{\frac{1}{n}}$.
4. $a \div a^{\frac{p}{q}}$.
5. $a^{\frac{4}{5}} \div a^{\frac{3}{4}}$.
6. $x^{\frac{7}{8}} \div x^{\frac{2}{3}}$.
7. $a^{\frac{3}{4}} \div a^{\frac{2}{3}}$.
8. $x^{\frac{2}{5}} \div x^{\frac{1}{7}}$.
9. $ab \div (ab)^{\frac{1}{7}}$.
10. $\left(\frac{1}{a}\right)^{\frac{5}{6}} \div \left(\frac{1}{a}\right)^{\frac{1}{2}}$.
11. $5^{\frac{2}{3}} \div 5^{\frac{1}{2}}$.
12. $x^{\frac{m}{n}} \div x^{\frac{p}{n}}$
13. $m^{\frac{1}{a}} \div m^{\frac{1}{b}}$.
14. $6^{\frac{2}{3}} \div 6^{\frac{1}{6}}$.
15. $m^{\frac{3}{8}} \div m^{\frac{1}{8}}$.
16. $p^{\frac{4}{5}} \div p^{\frac{3}{4}}$.

111. According to Law III (Sec. 102), *when an exponent is applied to a base having an exponent, the product of the exponents is the exponent of the result.*

For example:

$$(a^{\frac{1}{2}})^2 = a^{2 \cdot \frac{1}{2}} = a^1 = a.$$

$$(a^2)^{\frac{1}{4}} = a^{2 \cdot \frac{1}{4}} = a^{\frac{2}{4}} = a^{\frac{1}{2}}.$$

$$(a^{\frac{1}{3}})^{\frac{2}{5}} = a^{\frac{1}{3} \cdot \frac{2}{5}} = a^{\frac{2}{15}}.$$

A general formula for this statement is,

$$(a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{m}{n} \cdot \frac{p}{q}} = a^{\frac{mp}{nq}}.$$

ORAL EXERCISES

Simplify:

1. $(2^{\frac{1}{2}})^{\frac{1}{3}}$.
2. $(3^{\frac{1}{3}})^2$.
3. $(3^{\frac{2}{3}})^3$.
4. $(5^{\frac{3}{4}})^2$.
5. $(b^{\frac{1}{3}})^{\frac{1}{2}}$.
6. $(a^{\frac{1}{2}})^{\frac{1}{3}}$.
7. $(a^{\frac{2}{3}})^{\frac{5}{6}}$.
8. $(a^{\frac{3}{4}})^{\frac{2}{3}}$.
9. $(x^{\frac{5}{6}})^{\frac{7}{8}}$.
10. $(y^9)^{\frac{1}{3}}$.
11. $(5^{\frac{1}{2}})^{\frac{2}{3}}$.
12. $(3^{\frac{2}{3}})^{\frac{1}{3}}$.
13. $(a^{\frac{1}{5}})^{\frac{1}{2}}$.
14. $(a^{\frac{3}{4}})^4$.
15. $(10^{\frac{1}{6}})^{\frac{1}{3}}$.
16. $[(a+b)^{\frac{1}{2}}]^{\frac{1}{2}}$.
17. $[(a-b)^{\frac{1}{3}}]^{\frac{3}{4}}$.
18. $[(a^2-b^2)^{\frac{1}{4}}]^{\frac{1}{5}}$.
19. $[(x^n-y^n)^{\frac{1}{p}}]^{\frac{p}{q}}$.

112. According to Law IV (Sec. 102), *an exponent affecting a product is applied to each factor*, and according to Law V (Sec. 102), *an exponent affecting a fraction is applied to both numerator and denominator.*

For example:
$$(ab)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}.$$

$$(ab^{\frac{1}{2}}c^2)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{4}}c.$$

$$(8\,x^6y^2z)^{\frac{1}{3}} = 8^{\frac{1}{3}}x^{\frac{6}{3}}y^{\frac{2}{3}}z^{\frac{1}{3}} = 2\,x^2y^{\frac{2}{3}}z^{\frac{1}{3}}.$$

$$(a^m b^{\frac{m}{n}} c)^{\frac{p}{q}} = a^{\frac{mp}{q}} \cdot b^{\frac{mp}{nq}} \cdot c^{\frac{p}{q}}.$$

$$\left(\frac{x^2}{y^8}\right)^{\frac{1}{4}} = \frac{(x^2)^{\frac{1}{4}}}{(y^8)^{\frac{1}{4}}} = \frac{x^{\frac{2}{4}}}{y^{\frac{8}{4}}} = \frac{x^{\frac{1}{2}}}{y^2}.$$

A general formula for this statement is,

$$\left(\frac{a^{\frac{m}{n}}}{b^{\frac{p}{q}}}\right)^{\frac{r}{s}} = a^{\frac{mr}{ns}} \div b^{\frac{pr}{qs}}, \text{ or } \frac{a^{\frac{mr}{ns}}}{b^{\frac{pr}{qs}}}.$$

WRITTEN EXERCISES

Simplify:

1. $(a^3b^2)^{\frac{1}{2}}$.
2. $(a^2b^{\frac{1}{3}})^3$.
3. $(a^mb^n)^{\frac{1}{p}}$.
4. $(x^{\frac{1}{2}}y^{\frac{1}{4}})^3$.
5. $(a^{\frac{2}{3}}b^{\frac{1}{4}})^{\frac{1}{2}}$.
6. $(a^{\frac{3}{4}} \cdot b^{\frac{7}{8}})^8$.
7. $(86\,x^6y)^{\frac{1}{3}}$.
8. $\left(\frac{32\,a^5b^{10}}{c^{15}}\right)^{\frac{1}{5}}$.
9. $\left(\frac{64\,x^{12}}{y}\right)^{\frac{1}{6}}$.
10. $\left(\frac{9\,a^2b}{x}\right)^{\frac{1}{2}}$.
11. $\left(\frac{x^4y^2}{4}\right)^2$.
12. $(16\,x^4y^8)^{\frac{1}{4}}$.
13. $(a^6bn^{915})^{\frac{1}{3}}$.
14. $(27\,a^{12}b^9c^6)^{\frac{1}{3}}$.
15. $(m^{\frac{1}{2}}n^{\frac{1}{4}}p^{\frac{1}{5}})^{20}$.

113 When the bases are different and the fractional exponents are different, the exponents must have a common denominator, before any simplification by multiplication or division is possible.

For example: $a^{\frac{1}{2}}b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{\frac{2}{6}} = (a^3b^2)^{\frac{1}{6}}.$

A general formula for this statement is,

$$a^{\frac{m}{n}}b^{\frac{p}{q}} = a^{\frac{mq}{nq}} \cdot b^{\frac{np}{nq}} = (a^{mq} \cdot b^{np})^{\frac{1}{nq}}.$$

This is called *simplifying by reducing exponents to the same order.*

WRITTEN EXERCISES

Simplify by reducing the exponents to the same order:

1. $a^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
2. $a^{\frac{2}{3}} \cdot b^{\frac{3}{4}}$.
3. $v^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
4. $a^{\frac{1}{5}} \cdot b^{\frac{1}{4}}$.
5. $a^{\frac{1}{3}} \cdot b^{\frac{3}{4}}$.
6. $b^{\frac{1}{2}} \cdot 6^{\frac{1}{6}}$.
7. $5^{\frac{1}{5}} \cdot b^{\frac{1}{4}}$.
8. $a^{\frac{1}{2}} \div b^{\frac{1}{5}}$.
9. $m^{\frac{1}{3}} \cdot n^{\frac{1}{2}} \cdot p^{\frac{1}{4}}$.
10. $m^{\frac{2}{3}} \cdot n^{\frac{3}{5}} \div p^{\frac{1}{2}}$.
11. $p^{\frac{1}{n}} \cdot q^{\frac{1}{m}} \cdot r^{\frac{1}{3}}$.
12. $x^{\frac{p}{q}} \cdot y^{\frac{m}{n}} \cdot z^{\frac{1}{n}}$.

114. It is usually preferable to indicate roots by fractional exponents instead of by radical signs, since operations are thus more easily seen.

COMPARISON

	By Radicals	By Exponents
1.	$\sqrt{a}\sqrt[3]{a} = \sqrt[6]{a^3}\sqrt[6]{a^2} = \sqrt[6]{a^3a^2} = \sqrt[6]{a^5}$.	$a^{\frac{1}{2}}a^{\frac{1}{3}} = a^{\frac{3}{6}}a^{\frac{2}{6}} = a^{\frac{3}{6}+\frac{2}{6}} = a^{\frac{5}{6}}$.
2.	$\sqrt{b} \div \sqrt[3]{b} = \sqrt[6]{b^3} \div \sqrt[6]{b^2} = \sqrt[6]{b^3 \div b^2} = \sqrt[6]{b}$.	$b^{\frac{1}{2}} \div b^{\frac{1}{3}} = b^{\frac{3}{6}} \div b^{\frac{2}{6}} = b^{\frac{3}{6}-\frac{2}{6}} = b^{\frac{1}{6}}$.

WRITTEN EXERCISES

Simplify by use of fractional exponents as in the examples above:

1. $2^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}$.
2. $\sqrt{\frac{49}{121}}$.
3. $\sqrt{81a^{10}}$.
4. $\sqrt{5} \cdot \sqrt{75}$.
5. $3^{\frac{1}{2}} \cdot 2 \cdot 7^{\frac{1}{2}}$.
6. $3 \cdot 5^{\frac{1}{2}} \cdot 2\sqrt{3}$.

7. $\sqrt{8} \cdot 3\sqrt{2}$.

8. $2\sqrt{3} \cdot 3 \cdot \sqrt{10}$.

9. $2\sqrt[3]{3} \cdot 3\sqrt[3]{2}$.

10. $5^{\frac{1}{2}} \cdot 3 \cdot 5^{\frac{1}{3}}$.

11. $\sqrt{7} \cdot 11^{\frac{1}{2}}$.

12. $-8\sqrt{2} \cdot 12\sqrt{3}$.

13. $-\sqrt{12} \cdot 2\sqrt{3}$.

14. $a\sqrt{b} \cdot b\sqrt{a}$.

15. $(\sqrt{2}-\sqrt{3})2\sqrt{3}$.

16. $3^{\frac{1}{2}}(6^{\frac{1}{2}} - 2 \cdot 5^{\frac{1}{2}})$.

17. $(a\sqrt[3]{x})^2$.

18. $\sqrt{a^6b^8}$.

19. $\sqrt[3]{8a^5b^6}$.

20. $(27xy^2)^{\frac{1}{3}}$.

21. $(16x^4p)^{\frac{1}{4}}$.

22. $\sqrt[5]{32a^3b^{10}}$.

23. $\sqrt{36m^4n^2}$.

24. $\sqrt{64m^3n^6}$.

25. $(49a^4b^6)^{\frac{1}{2}}$.

26. $\sqrt[3]{2a^2b^2}$.

27. $\sqrt{32} \cdot \sqrt{2}$.

28. $(a^2b\sqrt[4]{xy^2})^3$.

29. $(m^3\sqrt[5]{p^4})^4$.

30. $\sqrt{6ab} \cdot \sqrt{2a}$.

31. $[2a(4a^2)^{\frac{1}{3}}]^2$.

32. $11^{\frac{1}{2}} \cdot 11^{\frac{1}{3}} \cdot 11^{\frac{1}{7}}$.

33. $\sqrt[8]{m^2\sqrt[3]{m^2}}$.

34. $\left(\dfrac{49a^4}{64b^6}\right)^{\frac{1}{2}}$.

35. $\sqrt[3]{a^3\sqrt{25b}}$.

36. $\sqrt{x^2y\sqrt[4]{25yz}}$.

37. $\sqrt[6]{a\sqrt[5]{ab^3}}$.

38. $\sqrt{\sqrt[3]{4x^2}}$.

39. $\sqrt{\sqrt[4]{8x^3}}$.

40. $\sqrt{\sqrt[5]{81a^4}}$.

41. $\sqrt[3]{x^2\sqrt[3]{9a^2}}$.

42. $\sqrt[3]{x^3y^6\sqrt[4]{27a^3}}$.

43. $a(a^2b)^{\frac{1}{3}} \cdot b(ab^2)^{\frac{1}{3}}$.

44. $3\sqrt{9a} \cdot 3^{\frac{1}{2}}$.

45. $5\sqrt[4]{a^3x^3} \cdot 2\sqrt[3]{ax}$.

46. $3\sqrt{8} \cdot 2\sqrt[3]{6} \cdot 3\sqrt[4]{54}$.

MEANING OF ZERO AND NEGATIVE EXPONENTS

115. The meaning of a^0 may be found as follows:

Assuming that Law I holds for a^0,
$$a^5 \cdot a^0 = a^{5+0} = a^5.$$

Dividing by a^5,
$$a^0 = \frac{a^5}{a^5} = 1.$$

That is:

Any number (not zero) with the exponent zero equals 1.

Thus, $5^0 = 1$, $10^0 = 1$, $(\frac{10}{3})^0 = 1$, $\left(\dfrac{a}{b}\right)^0 = 1$.

ORAL EXERCISES

State the value of:

1. $(ab)^0$.
2. $.9^0$.
3. 100^0.
4. $\left(\frac{a}{bc}\right)^0$.
5. $(-3)^0$.
6. $(\frac{1}{2})^0$.
7. $(-\frac{1}{27})^0$.
8. $3^2 \cdot 3^0$.
9. $3^2 \cdot 5^0$.
10. $\frac{1}{2} \cdot a^0$.
11. a^5a^0.
12. $a^m \cdot b^0$.
13. $a^5 \div a^0$.
14. $4^3 \div 4^0$.
15. $(\frac{1}{2})^0$.
16. $.9 \div 100^0$.
17. $9\frac{1}{2} \cdot 5^0$.
18. $3^0 \cdot 27^0$.
19. $(2^3 \cdot 3^2)^0$.

116. The meaning of the negative exponent may be found as follows:

Assuming that Law I holds for negative exponents,

$$5^{-3} \cdot 5^{+3} = 5^{-3+3} = 5^0 = 1.$$

That is, 5^{-3} is a multiplier such that its product with 5^{+3} is 1. But if the product of two numbers is 1, one is the reciprocal of the other.

Therefore, 5^{-3} is the reciprocal of 5^3 which is $\frac{1}{5^3}$.

Expressed in general terms:

$$a^{-n} \cdot a^n = a^{-n+n}$$
$$= a^0 = 1.$$
$$a^{-n} = \frac{1}{a^n}.$$

In words:

a^{-n} *means* $\frac{1}{a^n}$, *for all values of* n, *positive or negative, integral or fractional.*

ORAL EXERCISES

Find similarly the meaning of:

1. 4^{-3}.
2. 2^{-4}.
3. $(\frac{3}{4})^{-2}$.
4. $(-5)^{-6}$.
5. a^{-3}.
6. $a^{-\frac{3}{4}}$.
7. $(\frac{1}{2})^{-\frac{2}{3}}$.
8. $(-\frac{2}{3})^{-\frac{4}{5}}$.

State the value of:

9. $4^{-\frac{1}{2}}$.
10. $8^{-\frac{1}{3}}$.
11. $8^{-\frac{2}{3}}$.
12. $(.2^0)^{\frac{1}{2}}$.
13. $16^{-\frac{1}{4}}$.
14. $16^{-\frac{3}{4}}$.
15. $.125^{-\frac{1}{3}}$.
16. $.125^{-\frac{2}{3}}$.
17. $32^{-\frac{3}{5}}$.
18. $27^{-\frac{2}{3}}$.
19. $.36^{-\frac{3}{2}}$.
20. $64^{-\frac{5}{6}}$.
21. a^{-3}.
22. $(a^0)^{-2}$.
23. $a^{-\frac{3}{5}}$.
24. $a^{-\frac{1}{3}}$.

WRITTEN EXERCISES

Perform the operations indicated:

1. $(4^2 \cdot 5^4)^{-\frac{1}{2}}$.
2. $2^{14} \cdot 2^{-6}$.
3. $2^{\frac{2}{3}} \cdot 2^{-\frac{5}{6}}$.
4. $2^{-\frac{2}{3}} \cdot 2^{\frac{5}{6}}$.
5. $2^{-\frac{2}{3}} \cdot 2^{-\frac{5}{6}}$.
6. $3^{\frac{7}{8}} \cdot 3^{-\frac{1}{2}} \cdot 4^0$.
7. $\dfrac{5^{-\frac{1}{2}} \cdot 5^{\frac{3}{2}}}{5^0}$.
8. $10^{-5} \cdot 10^2 \cdot 10^0$.
9. $10^7 \cdot 10^{-7}$.
10. $27^{-\frac{3}{4}} \cdot 9^{\frac{3}{2}}$.
11. $\sqrt[4]{18^{-4}} \cdot \sqrt[3]{18^{-3}}$.
12. $\sqrt[3]{10^{-4}} \cdot \sqrt[5]{10^0}$.
13. $(10^{-4} \cdot 10^{-3})^{\frac{2}{7}}$.
14. $10^{-\frac{2}{3}} \cdot 10^{\frac{1}{2}}$.
15. $10^{\frac{1}{2}} \cdot 10^{\frac{5}{6}} \cdot 10^{\frac{1}{3}} \cdot (\frac{4}{7})^0$.
16. $(x^{-4})^2$.
17. $a^0 \cdot a^{\frac{1}{3}}$.
18. $\dfrac{a^{-\frac{1}{2}}}{a^{-\frac{1}{3}}}$ (or $a^{-\frac{1}{2}-(-\frac{1}{3})}$).
19. $a^{\frac{1}{2}} \cdot a^{-\frac{1}{2}}$ (or $a^{\frac{1}{2}-\frac{1}{2}}$).
20. $\dfrac{a^{\frac{1}{2}}}{a^{-\frac{1}{2}}}$ (or $a^{\frac{1}{2}-(-\frac{1}{2})}$).
21. $(a^{-\frac{1}{2}})^{-\frac{1}{3}}$.
22. $(x^{-4})^5$.

USE OF ZERO, NEGATIVE, AND FRACTIONAL EXPONENTS

117. We have defined zero and negative exponents so that Law I holds for them. It can be shown that the other four laws hold for these exponents as defined, but the laws will be applied here without proof.

118. The relation $a^{-n} \cdot a^n = a^0 = 1$ can be used to change the form of expressions.

I. **To free an expression from a negative exponent,** *multiply both numerator and denominator by a factor that will so combine with the factor having the negative exponent as to produce unity in accordance with the relation just mentioned. If more than one negative exponent is involved, apply the process for each.*

For example: $7^{-3} \cdot 2^4 = \dfrac{7^3 \cdot 7^{-3} \cdot 2^4}{7^3} = \dfrac{2^4}{7^3}$.

$$\frac{a^3}{b^{-5}} = \frac{b^5 \cdot a^3}{b^5 \cdot b^{-5}} = \frac{b^5a^3}{1} = b^5a^3.$$

$$\frac{x^{-2}}{t^{-5}} = \frac{t^5x^2 \cdot x^{-2}}{t^5x^2 \cdot t^{-5}} = \frac{t^5}{x^2}.$$

WRITTEN EXERCISES

Free from negative exponents:

1. $\dfrac{3^{-2}}{2^{-3}}$.
2. $\dfrac{a^{-5}}{b^{-3}}$.
3. $\dfrac{3^{-2}\cdot 4^{-1}}{5^{-3}}$.
4. $\dfrac{a^{-1}b^{-2}}{x^{-4}}$.
5. $\dfrac{x^{-a}}{a^{-x}}$.
6. $\dfrac{a^{-3}}{b^{3}}$.
7. $2a^{-\frac{1}{4}}$.
8. $\dfrac{ab}{c^{-5}}$.
9. $a^{-1}b^{-2}$.
10. $\dfrac{ax^{-4}}{b^{-3}x^{2}}$.
11. $\dfrac{5y^3}{5^{-1}y^{-4}}$.
12. $\dfrac{1}{a^{-3}x^{-3}}$.

II. **To free an expression from a fractional form,** *multiply both numerator and denominator by a factor that, in combination with the denominator, will produce unity. If more than one such form is involved, apply the process for each.*

For example:

$$\frac{a^3}{b^2}=\frac{b^{-2}a^3}{b^{-2}b^2}=b^{-2}a^3;$$

also,
$$\frac{2}{4^{-5}}+\frac{a^3}{x^2y^{-3}}=\frac{4^5\cdot 2}{4^5\cdot 4^{-5}}+\frac{x^{-2}y^3a^3}{x^{-2}y^3x^2y^{-3}}$$

$$=4^5\cdot 2+\frac{x^{-2}y^3a^3}{(x^{-2}x^2)(y^3y^{-3})}$$

$$=2\cdot 4^5+a^3x^{-2}y^3.$$

WRITTEN EXERCISES

Free from fractional forms:

1. $\dfrac{3}{5^2}$.
2. $\dfrac{1}{a^2b^3}$.
3. $\dfrac{a}{bx^3}$.
4. $\dfrac{a^{-2}}{t^{-3}}$.
5. $\dfrac{1}{5^2}+\dfrac{3}{2^3}$.
6. $\dfrac{x^2}{y^3}+\dfrac{y^2}{x^3}$.
7. $\dfrac{x}{y^{-1}}+\dfrac{a^2}{x^{-3}}$.
8. $\dfrac{a}{b}+\dfrac{a^{-1}}{b^{-1}}$.

III. **To transfer any specified factor from the numerator into the denominator, or vice versa,** *multiply the numerator and denominator by a factor that, in combination with the factor to be transferred, will produce unity.*

EXAMPLES

1. Transferring the factors of the denominator to the numerator:

$$\frac{x^7}{x^4y^{-3}} = \frac{x^{-4}y^3x^7}{x^{-4}y^3x^4y^{-3}} = x^{-4}y^3x^7 = x^3y^3.$$

2. Transferring the literal factors of the numerator to the denominator:

$$\frac{5\,a^3b^2}{4\,a^4b^{-3}} = \frac{5\,a^{-3}b^{-2}a^3b^2}{4\,a^{-3}b^{-2}a^4b^{-3}} = \frac{5}{4\,ab^{-5}}.$$

WRITTEN EXERCISES

In the following expressions:

(*a*) Transfer all literal factors to the numerator.

(*b*) Transfer all literal factors to the denominator.

1. $\frac{a^2x^3}{a^4x^7}$.

2. $\frac{6\,ay^2z^{-5}}{11\,a^3x^{-3}y^4}$.

3. $\frac{4\,a^2b^3c^5}{17\,a^3b^5c^{-3}}$.

4. $\frac{5\,a^{-\frac{1}{3}}b^{\frac{2}{5}}}{8\,a^{\frac{5}{6}}b^{-\frac{3}{5}}}$.

5. $\frac{a^{-\frac{2}{3}}b^{-5}c^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{-5}c^{-\frac{3}{4}}}$.

6. $\frac{7(a^{-7}b^{-5})^{\frac{1}{4}}}{6\,a^{-\frac{7}{4}}b^{-\frac{3}{2}}}$.

7. $\frac{5\,a^{-\frac{1}{3}}}{abx}$.

8. $\frac{6\,x^{-\frac{4}{5}}y^2}{5\,pq}$.

9. $\frac{a^{\frac{3}{4}}}{b^{\frac{7}{8}}}$.

119. The laws of exponents enable us to perform operations with polynomials containing fractional and negative exponents.

Thus:

$$(a^{\frac{2}{3}} + b^{\frac{1}{2}})^2 = (a^{\frac{2}{3}})^2 + 2\,a^{\frac{2}{3}}b^{\frac{1}{2}} + (b^{\frac{1}{2}})^2$$

$$= a^{\frac{4}{3}} + 2\,a^{\frac{2}{3}}b^{\frac{1}{2}} + b.$$

WRITTEN EXERCISES

Perform the indicated operations:

1. $(a^{\frac{3}{2}}-b^{\frac{5}{4}})^2$.
2. $(4\,a^{\frac{2}{3}}x^{\frac{1}{2}}+2^3)^2$.
3. $(a^{-\frac{1}{2}}+b^{\frac{3}{2}})^2$.
4. $(a^n-3\,b^r)^2$.
5. $(x^{-\frac{3}{4}}-y^{-\frac{2}{3}})(x^{-\frac{3}{4}}+y^{-\frac{2}{3}})$.
6. $(x^{-n}+1)(x^{-n}-1)$.
7. $(x^{\frac{6}{5}}+3)(x^{\frac{6}{5}}+5)$.
8. $\dfrac{a^5-x^4}{a^{\frac{5}{2}}-x^2}$.
9. $(x^n-y^n)^2$.
10. $(a^{2n}b^{3r}-1)^2$.
11. $\dfrac{a^p\cdot b^q}{a^{p+1}\cdot b^{q-1}}$.
12. $(a^n+t^n)^2$.
13. $(a^{\frac{1}{3}}b^{\frac{3}{2}}+x^{-\frac{5}{2}})(a^{\frac{1}{3}}b^{\frac{3}{2}}-x^{-\frac{5}{2}})$.

Express as a product of two factors:

14. x^6-m^{-4}.
15. $a^8-2\,a^4x^{\frac{1}{3}}+x^{\frac{2}{3}}$.
16. $a^{2n}+2\,a^nb^n+b^{2n}$.
17. $x^{4n}-4\,x^{2n}y^{2n}+4\,y^{4n}$.
18. $y^{-7}-x^{-10}$.
19. x^r-4.
20. $1+8\,x^{-\frac{5}{2}}+16\,x^{-5}$.
21. $x^{12}+6\,x^6y^{-\frac{7}{2}}+9\,y^{-7}$.

SUMMARY

I. **Definitions.**

Meaning of the Fractional Exponents. $a^{\frac{p}{q}}$ denotes the qth root of the pth power, or the pth power of the qth root of a. Sec. 107.

Meaning of the Exponent Zero. Any number (not 0) with the exponent zero equals 1. Sec. 115.

Meaning of Negative Exponents. a^{-n} means $\dfrac{1}{a^n}$ for all values of n, positive or negative, integral or fractional. Sec. 116

II Laws of Exponents.

1. $a^m \cdot a^r = a^{m+r}$. Sec. 99.

2. $a^m \div a^r = a^{m-r}$. Sec. 100.

3. $(a^m)^r = a^{mr}$. Sec. 101.

4. $(ab)^n = a^n b^n$. Sec. 101.

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. Sec. 101.

These laws apply for all values of the exponents, m, n, r, positive, negative, integral, or fractional.

III. Processes with Exponents.

1. When the bases of the factors are the same, the exponent of the product is found by adding the given exponents (Law I). Sec. 109.

2. When the bases of the expressions are the same, the exponent of the quotient is found by subtracting the exponent of the divisor from the exponent of the dividend (Law II). Sec. 110.

3. When an exponent is applied to a number having an exponent, the product of the exponents is taken as the exponent of the result (Law III). Sec. 111.

4. An exponent affecting a product is applied to each factor (Law IV). Sec. 112.

5. An exponent affecting a fraction is applied to both numerator and denominator (Law V). Sec. 112.

6. When the bases are different and the fractional exponents are different, the exponents must have, or be made to have, a common denominator, before any simplification by multiplication or division is possible. Sec. 113.

7. To free an expression from a negative exponent, multiply both numerator and denominator by a factor that will so combine with the factor having the negative exponent as to produce unity, in accordance with the relation $a^{-n} \cdot a^n = a^0$. Sec. 118.

8. To free an expression from the fractional form, multiply numerator and denominator by a factor that, in combination with the denominator, will produce unity. If more than one such form is involved, apply the process for each. Sec. 118.

9. To transfer any specified factor from the numerator into the denominator, or vice versa, multiply the numerator and denominator by a factor that, in combination with the factor to be transferred, will produce unity. Sec. 118.

REVIEW

WRITTEN EXERCISES

Express with positive exponents:

1. $m^{-\frac{3}{4}}n^8$. 2. $4x^{-\frac{1}{2}}y^{-1}z$. 3. $3a^{-5}b^5$. 4. $17x^{-\frac{5}{2}}y^{-7}z^{-\frac{7}{8}}$.

Transfer all literal factors from the denominator to the numerator:

5. $\dfrac{x^{\frac{3}{4}}}{x^{-\frac{2}{3}}}$. 6. $\dfrac{ab}{a^{-\frac{1}{7}}b^{-3}}$. 7. $\dfrac{1}{6x^{-2}y^{\frac{3}{2}}}$. 8. $\dfrac{5}{ay^{-4}}$.

Multiply:

9. $(2+\sqrt{x+1})^2$.

10. $\sqrt{5}\cdot\sqrt[3]{6}$.

11. $5\sqrt[4]{m^{-3}}\cdot 2m^{-1}$.

12. $p\cdot p^{-\frac{3}{4}}$.

13. $(a^{-1}-b^{-1})(a^{-\frac{1}{2}}-b^{-\frac{1}{2}})$.

14. $(x^2-1)(x^{\frac{1}{2}}+1)$.

Remove the parentheses:

15. $(a^{-\frac{3}{4}})^2$.

16. $(\sqrt[3]{a^{-1}})^3$.

17. $\left(\dfrac{1}{\sqrt[4]{p^3}}\right)^{\frac{4}{5}}$.

18. $\left(\dfrac{1}{n^{-2}}\right)^5$.

19. $\left(a^{\frac{5n}{6}}\right)^{\frac{3}{10n}}$.

20. $\left(x^{\frac{m^2}{n^2}-1}\right)^{\frac{n}{m-n}}$.

21. $\left(\dfrac{1}{x^{-\frac{p}{q}}}\right)^{\frac{q}{p}}$.

22. $\left(m^{\frac{pq}{xy}}\right)^{\frac{x^2y}{pq}}$.

23. $\left(s^{\frac{1}{n-1}}\right)^{n^2-1}$.

SUPPLEMENTARY WORK

ADDITIONAL EXERCISES

Perform the operations indicated:

1. $(8\,a^3b^{\frac{1}{2}}c^{-\frac{3}{4}})^{\frac{2}{3}}$.
2. $(16\,a^{\frac{2}{3}}bc^{-4})^{\frac{3}{4}}$.
3. $(5\,ax^3y^2)^{\frac{1}{2}}$.
4. $(\sqrt[3]{4\,x^2})^{\frac{1}{4}}$.
5. $(64\,x^{-3}y^{\frac{1}{2}})^{-\frac{1}{6}}$.
6. $(-\,250\,x^3y^{-\frac{3}{5}}z^{-3})^{\frac{2}{3}}$.
7. $(-243\,a^{-\frac{1}{2}}y^{-1}z^{\frac{1}{4}})^{-\frac{4}{5}}$.
8. $\sqrt[p-q]{x^{p^2-q^2}}$.
9. $\sqrt[p]{x^{-2p}y^{-p^2}z^{-p}}$.
10. $\sqrt{a^{-m}b^{4m}}$.
11. $3\,a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-1}\cdot 2\,a^{\frac{1}{4}}b^{\frac{1}{2}}c$.
12. $10\,ab^{-\frac{1}{2}}c^{\frac{3}{4}}\div 2\,a^{-1}b^2z^{\frac{1}{2}}$.
13. $a^mb^{-2n}c^{-2}\div ab^nz^{-3}$.
14. $(3\,a^{-2}\div b^{-2})^{-5}$.
15. $(a^{p-q})^{p+q}\cdot a^{p^2}a^{q^2}$.
16. $\dfrac{\sqrt{a}}{\sqrt[5]{a^2b^3}}\cdot\dfrac{\sqrt[4]{a^3b^2}}{\sqrt[10]{a^7b^9}}$.
17. $x^{-3}\cdot x^{\frac{3}{4}}\cdot x^2\cdot\sqrt[5]{x}$.
18. $\dfrac{\sqrt[3]{x}}{\sqrt{x}}\cdot\dfrac{x^3}{x}\cdot\dfrac{\sqrt[3]{x^2}}{x^2}$.
19. $[\{(a^2-b^{-2})^{-1}\}^{-2}]^5$.
20. $[(a^{-\frac{1}{2}})^{\frac{2}{3}}]^{-12}$.
21. $\left\{\sqrt{ab^{-2}\sqrt{ab}}\right\}^4$.
22. $\{(a^{-3}b^2)^{\frac{1}{2}}\}^{-\frac{2}{3}}$.
23. $\sqrt[3]{a^2\sqrt{a^{-1}}}$.
24. $[(x^a)^{-b}]^{-\frac{1}{a}}\div[(x^{-b})^c]^{-\frac{1}{c}}$.
25. $-a^3b^{-4}c^{-3}d^5\cdot -a^{-2}b^5c^4d^{-5}$.
26. $a^2x-3\,y^6\cdot a^3x^6y^9$.
27. $3\,x^{-\frac{1}{2}}5\,x^{\frac{4}{5}}\cdot 10\,x^{-\frac{1}{2}}$.
28. $\sqrt{a^{-1}\sqrt{a^3\sqrt{a-4}}}$.
29. $\{x^{-\frac{3}{2}}y(xy^{-2})^{-\frac{1}{2}}(x^{-1}y)^{\frac{2}{3}}\}^{\frac{3}{2}}$.

Simplify:

30. $$\frac{\dfrac{x-y}{2\,y+x}+\dfrac{1}{2}}{3\frac{1}{2}-\left(\dfrac{x+2\,y}{5\,x+7\,y}\right)^{-1}}.$$

31. $\left(x^{\frac{n+1}{n-1}}\div x^{\frac{n-1}{n+1}}\right)^{\frac{n-1}{2n}}$.

32. $[(a^{p+q})^{p-q}(a^{q^3})^{\frac{1}{q}}]^{\frac{1}{p^2}}$.

33. (a) $\dfrac{\sqrt{(2^{\frac{1}{3}}+2^{-\frac{1}{3}})^2-4}}{4(2^{\frac{1}{3}}-2^{-\frac{1}{3}})} \cdot \dfrac{5+\sqrt{21}}{5-\sqrt{21}}.$

(b) $\sqrt[7]{x^2y^{12}}\left(\dfrac{1}{xy}\right)^{\frac{1}{7}}\left(\dfrac{y^2}{x}\right)^{-\frac{2}{7}}.$

(c) $3\sqrt{\dfrac{5}{2}}+\sqrt{40}+\sqrt{\dfrac{2}{5}}-\dfrac{1}{\sqrt{10}}.$

34. $\frac{1}{3}\sqrt{45}+4\sqrt{\frac{5}{4}}-\sqrt{125}.$

35. Express the product $\sqrt[3]{a^2}\sqrt{a^3}$ as a single radical.

36. Divide $2x^{\frac{9}{2}}y^{-3}-5x^{\frac{7}{2}}y^{-2}+7x^{\frac{5}{2}}y^{-1}-5x^{\frac{3}{2}}+2x^{\frac{1}{2}}y$ by $x^3y^{-3}-x^2y^{-2}+xy^{-1}.$

Find equivalent expressions with rational divisors:

37. $3\sqrt{2a}\div 2\sqrt{3b}.$

38. $b\sqrt{a^2}\div\sqrt{ab}.$

39. $\sqrt{40x^3y}\div x\sqrt{5y}.$

40. $x\sqrt[3]{y}\div y\sqrt[3]{x}.$

41. $2\sqrt[3]{2a^2}\div\sqrt[3]{4a}.$

42. $-\frac{3}{4}\sqrt{\frac{3}{5}}\div\frac{3}{10}\sqrt{3}.$

43. $6\sqrt[3]{54x^2}\div 2\sqrt[3]{2x^2}.$

44. $a^2\sqrt[4]{48ab^3}\div 2ab\sqrt[4]{3ab^2}.$

45. $4ax\div\sqrt[4]{ax}.$

46. $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}.$

47. $\dfrac{1}{a-\sqrt{a^2-x^2}}.$

48. $\dfrac{(\sqrt{5}-2)(3+\sqrt{5})}{5-\sqrt{5}}.$

49. $\dfrac{p-\sqrt{q}}{p+\sqrt{q}}.$

50. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}.$

51. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}.$

52. $\dfrac{\sqrt{x}-\sqrt{x+y}}{\sqrt{x}+\sqrt{x+y}}.$

53. $\dfrac{\sqrt{a^2+b^2}+b}{\sqrt{a^2+b^2}-b}.$

54. $\dfrac{3\sqrt{x-3}+\sqrt{x+3}}{3\sqrt{x-3}-\sqrt{x+3}}.$

55. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$.

56. $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}$.

57. $\dfrac{1}{2+\sqrt{5}-\sqrt{2}}$.

Suggestion. Rationalize the denominator in two steps, using as first factor

$$2-(\sqrt{5}-\sqrt{2}).$$

Solve:

58. $3x+\sqrt{x^2-2x+5}=1$.

59. $\dfrac{\sqrt{x}+a}{b}=\dfrac{x}{a^2-2ab+b^2}$.

60. $\dfrac{x}{a+b}+\sqrt{(a+b)^2+ab-x}=a+b+\dfrac{2ab}{a+b}$.

61. $\sqrt{x}(2a-b+\sqrt{x})=3a^2-ab$.

CHAPTER V

LOGARITHMS

MEANING AND USE OF LOGARITHMS

120. Use of Exponents in Computation. By applying the laws of exponents certain mathematical operations may be performed by means of simpler ones. The following table of powers of 2 may be used in illustrating some of these simplifications:

$1 = 2^0$	$32 = 2^5$	$1024 = 2^{10}$	$32768 = 2^{15}$
$2 = 2^1$	$64 = 2^6$	$2048 = 2^{11}$	$65536 = 2^{16}$
$4 = 2^2$	$128 = 2^7$	$4096 = 2^{12}$	$131072 = 2^{17}$
$8 = 2^3$	$256 = 2^8$	$8192 = 2^{13}$	$262144 = 2^{18}$
$16 = 2^4$	$512 = 2^9$	$16384 = 2^{14}$	$524288 = 2^{19}$

121. Application of Law I, Sec. 99, p. 53.

EXAMPLES

1. Find: $8 \cdot 32$.

From the table, $8 = 2^3$, (1)

and $32 = 2^5$. (2)

Then, $8 \cdot 32 = 2^3 \cdot 2^5 = 2^8$, (3)

and, according to the table, $2^8 = 256$. (4)

2. Find: $2048 \cdot 64$.

From the table, $2048 = 2^{11}$, (1)

and $64 = 2^6$. (2)

Then, $2048 \cdot 64 = 2^{11} \cdot 2^6 = 2^{17}$, (3)

and, according to the table, $2^{17} = 131072$. (4)

Thus the process is simply one of inspection. In the above example we merely added 11 and 6 and looked in the table for the number opposite to 2^{17}.

ORAL EXERCISES

State the following products by reference to the table:

1. $16 \cdot 256$.
2. $32 \cdot 128$.
3. $64 \cdot 512$.
4. $8 \cdot 2048$.
5. $32 \cdot 32$.
6. $64 \cdot 64$..
7. $32 \cdot 2048$.
8. $16 \cdot 4096$.
9. $128 \cdot 512$.
10. $128 \cdot 1024$.
11. $8 \cdot 16384$.
12. $32 \cdot 4096$.

122. Application of Law II, Sec. 100, p. 54.

EXAMPLES

1. Find: $\dfrac{256}{32}$.

From the table, $256 = 2^8$, (1)

and $32 = 2^5$. (2)

Hence, $\dfrac{256}{32} = \dfrac{2^8}{2^5} = 2^{8-5} = 2^3$, (3)

and, according to the table, $2^3 = 8$. (4)

2. Find: $\dfrac{65536}{2048}$.

As above, $\dfrac{65536}{2048} = \dfrac{2^{16}}{2^{11}} = 2^5 = 32$.

ORAL EXERCISES

By use of the table determine the value of the following:

1. $\dfrac{1024}{128}$.
2. $\dfrac{8192}{64}$.
3. $\dfrac{32768}{1024}$.
4. $\dfrac{64 \cdot 512}{16 \cdot 128}$.
5. $\dfrac{32 \cdot 2048}{512}$.
6. $\dfrac{128 \cdot 131072}{64 \cdot 8 \cdot 8192}$.

123. Application of Law III, Sec. 101, p. 54.

EXAMPLES

1. Find: 16^3.

By the table, $16 = 2^4$. (1)

Hence, $16^3 = (2^4)^3 = 2^{12}$. (2)

and, according to the table, $2^{12} = 4096$. (3)

2. Find: $\sqrt{1024}$.

As above, $1024 = (1024)^{\frac{1}{2}} = (2^{10})^{\frac{1}{2}} = 2^5 = 32$, according to the table.

3. Find: $\sqrt[5]{32768}$.

As above, $\sqrt[5]{32768} = (2^{15})^{\frac{1}{5}} = 2^3 = 8$.

ORAL EXERCISES

By use of the table find the value of:

1. 32^3.	4. 64^2.	7. $\sqrt{8192}$.	10. $512^{\frac{2}{3}}$.
2. 3^5.	5. 256^2.	8. $\sqrt[3]{4096}$.	11. $\sqrt[3]{32768}$.
3. 32^5.	6. 16^4.	9. $\sqrt[4]{65536}$.	12. $\sqrt[5]{1024}$.

124. The examples and exercises above show that the laws of exponents furnish a powerful and remarkably easy way of making certain computations.

In the above illustrations we have used a table based on the number 2, and have limited the table to integral exponents; but for practical purposes a table based on 10 is used and is made to include fractional exponents.

For example:

1. It is known that approximately,

$$2 = 10^{\frac{3}{10}} \text{ or } 10^{.3} \text{ (more accurately } 10^{.301}).$$

From this we can express 20 as a power of 10, for

$$20 = 10 \cdot 2 = 10^1 \cdot 10^{.301} = 10^{1.301}.$$

Similarly, $$200 = 10 \cdot 20 = 10^1 \cdot 10^{1.301} = 10^{2.301},$$

and $$2000 = 10 \cdot 200 = 10^1 \cdot 10^{2.301} = 10^{3.301}.$$

2. It is known that approximately $763 = 10^{2.88}$.

Then $$7630 = 10 \cdot 763 = 10^1 \cdot 10^{2.88} = 10^{3.88},$$

and $$76300 = 100 \cdot 763 = 10^2 \cdot 10^{2.88} = 10^{4.88}.$$

Similarly, $$76.3 = \frac{763}{10} = \frac{10^{2.88}}{10^1} = 10^{2.88-1} = 10^{1.88},$$

and $$7.63 = \frac{763}{100} = \frac{10^{2.88}}{10^2} = 10^{2.88-2} = 10^{0.88}.$$

WRITTEN EXERCISES

Given $48 = 10^{1.68}$; express as a power of 10:

1. 480. 2. 4800. 3. 48,000. 4. 4.8.

Given $649 = 10^{2.81}$; express as a power of 10:

5. 6490. 7. 649,000. 9. 6.49.
6. 64,900. 8. 64.9. 10. 649,000,000.

Given $300 = 10^{2.47}$; express as a power of 10:

11. 3. 13. 3000. 15. 300,000.
12. 30. 14. 30,000. 16. 3,000,000.

125. The use of the base 10 has several advantages.

I. The exponents of numbers not in the table can readily be found by means of the table.

To make this clear, let us suppose that a certain table expresses all integers from 100 to 999 as powers of 10; then 30, although not in this table, can be expressed as a power of 10 by reference to the table.

For, $30 = \frac{300}{10}$, and since 300 is in the supposed table we may find by reference to the table that $300 = 10^{2.47}$, and hence, $30 = \frac{10^{2.47}}{10^1} = 10^{1.47}$.

Similarly, 3.76 is not in the supposed table, but 376 is and $3.76 = \frac{376}{100} = \frac{376}{10^2}$. Therefore it is necessary only to subtract 2 from the power of 10 found for 376 in order to find the power of 10 equal to 3.76.

Similarly, 4680 is not in the table, but 468 is and $4680 = 468 \cdot 10^1$. Therefore, it is necessary only to add 1 to the power of 10 found for 468 in order to find the power of 10 equal to 4680.

Such a table would not enable us to express in powers of 10 numbers like 4683, 46.83, and 356,900, but only numbers of 3 or fewer digits, which may be followed by any number of zeros.

Similar conditions would apply to a table of powers for numbers from 1000 to 9999, from 10,000 to 99,999, and so on.

II. The integral part of the exponent can be written without reference to a table.

For example:

1. 879 is greater than 100, which is the second power of 10, and less than 1000, or the third power of 10. That is, 879 is greater than 10^2 but less than 10^3. Therefore the exponent of the power of 10 which equals 879 is 2. + a decimal.

2. Similarly, 87.9 lies between 10 and 100, or between 10^1 and 10^2, hence the exponent of the power of 10 that is equal to 87.9 is 1. + a decimal.

ORAL EXERCISES

State the integral part of the exponent of the power of 10 equal to each of the following:

1. 35.	4. 25.	7. 25.5.
2 350.	5. 2500.	8. 365.5.
3. 36.5.	6. 36,500.	9. 17.65.

III. If two numbers have the same sequence of digits but differ in the position of the decimal point, the exponents of the powers of 10 which they equal have the same decimal part.

For example:

Given that $274.3 = 10^{2.43}$,

we have $27.43 = \frac{274.3}{10} = \frac{10^{2.43}}{10^1} = 10^{1.43}$,

also $2743 = 10 \cdot 274.3 = 10^1 \cdot 10^{2.43} = 10^{3.43}$,

also $274{,}300 = 1000 \cdot 274.3 = 10^3 \cdot 10^{2.43} = 10^{5.43}$.

In each instance the decimal part of the exponent is the same. It is evident that this will be the case in all similar instances, for shifting the decimal point is equivalent to multiplying or dividing repeatedly by 10, which is equivalent to changing the integral part of the exponent by adding or subtracting an integer.

ORAL EXERCISES

Given $647 = 10^{2.81}$, state the decimal part of the exponent of the power of 10 that equals:

1. 64.7.	3. 6470.	5. 647,000.
2. 6.47.	4. 64,700.	6. 6,470,000.

Given $568.1 = 10^{2.75}$, state the decimal part of the exponent of the power of 10 that equals:

7. 56.81.
8. 5.681.
9. 5681.
10. 56,810.
11. 568,100.
12. 56,810,000.

126. Logarithms. Exponents when used in this way for computation are called **logarithms**, abbreviated **log**.

127. The number to which the exponents are applied is called the **base**.

For the purposes of computation the base used is 10.

According to the above definition the equation $30 = 10^{1.48}$ may be written $\log 30 = 1.48$, which is read "The logarithm of 30 is 1.48." These equations mean the same thing; namely, that 1.48 is (approximately) the power of 10 that equals 30.

WRITTEN EXERCISES

Write the following in the notation of logarithms:

1. $700 = 10^{2.84}$.
2. $75 = 10^{1.87}$.
3. $6 = 10^{0.77}$.
4. $50 = 10^{1.69}$.
5. $361 = 10^{2.55}$.
6. $45 = 10^{1.65}$.

Write the following as powers of 10:

7. $\log 20 = 1.3$.
8. $\log 500 = 2.70$.
9. $\log 3 = 0.47$.
10. $\log 7 = 0.84$.
11. $\log 111 = 2.04$.
12. $\log 21 = 1.32$.

128. Characteristic and Mantissa. The integral part of a logarithm is called its **characteristic**, and the decimal part its **mantissa**.

ORAL EXERCISES

1–12. State the characteristic and the mantissa in each of the logarithms in Exercises 1–12 above.

129. According to Sec. 125, II (p. 77), the characteristics of logarithms can be determined by inspection, consequently tables of logarithms furnish only the mantissas.

EXPLANATION OF THE TABLES

130. The use of the tables, pp. 84 and 85, is best seen from an example.

Find the logarithm of 365.

The first column in the table, p. 84, contains the first two figures of the numbers whose mantissas are given in the table, and the top row contains the third figure.

Hence, find 36 in the left-hand column, p. 84, and 5 at the top.

In the column under 5 and opposite to 36 we find 5623, the required mantissa.

Since 365 is greater than 100 (or 10^2) but less than 1000 (or 10^3), the characteristic of the logarithm is 2.

Therefore, log 365 is 2.5623.

WRITTEN EXERCISES

By use of the table find the logarithms of :

1. 25.	5. 99.	9. 9.9.	13. 1000.
2. 36.	6. 86.	10. 8.6.	14. 5000.
3. 50.	7. 999.	11. 33,000.	15. 505.
4. 75.	8. 800.	12. 99,900.	16. 5.05.

131. Negative Characteristics. An example will serve to show how negative characteristics arise:

From $\log 346 = 2.5391$, we find,

$$\log 34.6 = \log \frac{346}{10} = \log 346 - \log 10 = 2.5391 - 1 = 1.5391.$$

$$\log 3.46 = \log \frac{34.6}{10} = 1.5391 - 1 = 0.5391.$$

$$\log .346 = \log \frac{3.46}{10} = 0.5391 - 1.$$

In the last line we have a positive decimal less 1, and the result is a negative decimal; viz. $-.4609$. But to avoid this change of mantissa, it is customary not to carry out the subtraction, but simply to indicate it. It might be written $-1 + .5391$, but it is customarily abridged into $\bar{1}.5391$. The mantissa is kept positive in all logarithms. The logarithm $\bar{1}.5391$ says that the corresponding number is greater than 10^{-1} (or $\frac{1}{10}$), but less than 10^0 or 1.

We now write $\log\ .346 = \bar{1}.5391$.

Similarly, $\log .0346 = \log \frac{.346}{10} = \bar{1}.5391 - 1 = \bar{2}.5391.$

Thus we see that the mantissa remains the same, no matter how the position of the decimal point is changed. The mantissa is determined solely by the sequence of digits constituting the number.

The characteristic is determined solely by the position of the decimal point. Negative characteristics, like positive ones, are determined by inspection.

EXAMPLES

1. What is the characteristic of log .243?

.243 is more than .1 or 10^{-1}, but less than 1 or 10^0. Hence,

$$.243 = 10^{-1+\text{ a decimal}}.$$

The characteristic is $\bar{1}$.

2. Find the characteristic of log .0593.

.0593 is more than .01 or 10^{-2}, but less than .1 or 10^{-1}. Hence,

$$.0593 = 10^{-2+\text{ a decimal}}.$$

The characteristic is -2, or $\bar{2}$.

3. Similarly, since .00093 is greater than .0001 or 10^{-4}, but less than .001 or 10^{-3},

$$\log .00093 = \bar{4} + \text{a decimal}.$$

132. The characteristic having been determined, the mantissa is found from the table in the usual way.

For example:

$$\log\ .243 = \bar{1}.3856,$$
$$\log\ .0593 = \bar{2}.7731,$$
$$\log .00093 = \bar{4}.9685.$$

WRITTEN EXERCISES

Find the logarithms of:

1. .35.	3. .105.	5. .0023.	7. .00342.
2. .634.	4. .027.	6. .0123.	8. .0004.

133. In finding the number corresponding to a logarithm with negative characteristic, the same method is followed as when the characteristic is positive. The mantissa determines the sequence of digits constituting the number; the characteristic fixes the position of the decimal point.

For example :

If $\log n = \bar{2}.5955$, the digits of n are 394. The characteristic $\bar{2}$ says that n is greater than 10^{-2} (or .01), but less than 10^{-1} (or .1). Hence, the decimal point must be so placed that n has no tenths but some hundredths. Therefore $n = .0394$.

WRITTEN EXERCISES

By use of the tables find the numbers whose logarithms are:

1. $\bar{1}.6232$.	7. $\bar{2}.0792$.	13. $\bar{3}.0969$.
2. $\bar{1}.4914$.	8. $\bar{2}.7076$.	14. $\bar{1}.6972$.
3. 2.4281.	9. 4.9196.	15. 3.9284.
4. 1.9196.	10. 3.2201.	16. 2.9284.
5. 0.9196.	11. 0.2201.	17. 1.9284.
6. 3.4281.	12. 1.2201.	18. 5.7832.

Find n if:

19. $\log n = 1.9289$.

20. $\log n = 0.9289$.

21. $\log (n - 1) = 3.9294$.

22. $\log (\frac{1}{2} n) = 1.6128$.

USE OF THE TABLES FOR COMPUTATION

134. For use in computation by logarithms the laws of exponents may be expressed thus:

I. $10^m \cdot 10^r = 10^{m+r}$. *The logarithm of a product is the sum of the logarithms of the factors.*

II. $\dfrac{10^m}{10^r} = 10^{m-r}$. *The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.*

III. $(10^m)^r = 10^{mr}$. *The logarithm of a number with an exponent is the product of the exponent and the logarithm of the number.*

Since r may be positive or negative, integral or fractional, Law III provides not only for raising to integral powers, but also for finding reciprocals of such powers, and for extracting roots.

EXAMPLES

1. Multiply 21 by 37.

   1. $\log 21 = 1.3222$, table, p. 84.
   2. $\log 37 = 1.5682$, table, p 84.
   3. Adding, $\log 21 + \log 37$, or $\log (21 \times 37) = 2.8904$ (Sec. 134).
   4. $\therefore\ 21 \times 37 = 777$ from table, p. 85.

2. Divide 814 by 37.

   1. $\log 814 = 2.9106$, table, p. 85.
   2. $\log 37 = 1.5682$, table, p. 84.
   3. $\therefore\ \log 814 - \log 37 = 2.9106 - 1.5682 = 1.3424$.
   4. $\therefore\ 814 \div 37 = 22$, table, p. 84.

3. Extract the cube root of 729.

   1. $\sqrt[3]{729} = (729)^{\frac{1}{3}}$.
   2. $\log 729^{\frac{1}{3}} = \frac{1}{3} \log 729$ (Sec. 134).
   3. $\log 729 = 2.8627$, table, p. 85.
   4. $\frac{1}{3}$ of $2.8627 = 0.9542$.
   5. $\therefore\ (729)^{\frac{1}{3}}$ or $\sqrt[3]{729} = 9$, table, p. 85.

Compute by use of logarithms:

1. 8×15.	6. 5^4.	11. 19^2.
2. 41×23.	7. 31^2.	12. 7^3.
3. 37×17.	8. $\sqrt{414}$.	13. 4^5.
4. 12×17.	9. $\sqrt[3]{343}$.	14. $\sqrt{196}$.
5. $893 \div 19$.	10. $940 \div 47$.	15. $\sqrt[3]{216}$.

Since the logarithm is approximate, the result in general is approximate. Thus, $\log \sqrt[4]{(256)} = 0.60205$, which is not the logarithm of 4, but is sufficiently near to be recognized in the table.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

135. How to calculate the logarithm of a number not found in the table is best seen from an example.

EXAMPLE

Find the logarithm of 257.3 (see 16th line of table, p. 84):

	0	1	2	3	4	5	6	7	8	9
25	3979	3997	4014	4031	4048	4065	4082	**4099**	**4116**	4133

The numbers nearest to 257.3 whose logarithms are given in the table are 257 and 258. We have

$$\begin{array}{r} \log 258 = 2.4116 \\ \underline{\log 257 = 2.4099} \\ \text{Difference,}\quad .0017 \end{array}$$

That is, an increase of 1 in the number causes an increase of .0017 in the logarithm. Assuming that an increase of .3 in the number would cause an increase of .3 of .0017, or .0005, in the logarithm, then

$$\log 257.3 = 2.4099 + .0005 = 2.4104.$$

NOTES: 1. The product of .3 and .0017 is .00051, but we take only four places, because the mantissas as given in the table are expressed to four places only. If the digit in the fifth decimal place of the correction is more than 5, we replace it by a unit in the fourth place.

2. The difference between two succeeding mantissas of the table (called the **tabular difference**) can be seen by inspection.

3. What is written in finding the logarithm of 257.3 should be at most the following:

$$\begin{array}{rr} \log 257 = & 2.4099 \\ \underline{\text{correction for } .3 =} & \underline{5} \\ \log 257.3 = & 2.4104 \end{array} \qquad \begin{array}{rr} \text{tabular difference} & 17 \\ & \underline{.3} \\ & 5.1 \end{array}$$

4. The corrections are made on the assumption that the change in the logarithm is proportional to the change in the number. This is sufficiently accurate when used within the narrow limits here prescribed.

WRITTEN EXERCISES

Find the logarithm of:

1. 1235.	5. 1425.	9. 3.142.	13. .4071.	17. .3002.
2. 23.5.	6. 1837.	10. 1.414.	14. 85.51.	18. 9009.
3. 2.36.	7. 6720.	11. 1.732.	15. .0125.	19. 12.02.
4. .0237.	8. 67.25.	12. .6226.	16. .4267.	20. 5.008.

136. To calculate the number whose logarithm is given apply the table as follows:

(1) If the given logarithm is in the table, the number can be seen at once.

(2) If the given logarithm is not in the table, the number corresponding to the nearest logarithm of the table may be taken.

A somewhat closer approximation may be found by using the method of the following example:

EXAMPLE

Find the number whose logarithm is 1.4271. The mantissas nearest to this are found in the 17th line of table, p. 84.

	0	1	2	3	4	5	6	7	8	9
26	4150	4166	4183	4200	4216	4232	4249	**4265**	**4281**	4298

The process is the reverse of that of finding the logarithm.

The next smaller mantissa in the table is 4265, corresponding to the number 267. The difference between this mantissa and the given mantissa is 6. The tabular difference between 4265 and the next larger mantissa is 16. An increase of 16 in the logarithm corresponds to an increase of 1 in the number. Hence, an increase of 6 in the logarithm corresponds to an increase of $\frac{6}{16}$ of 1, or .4, in the number. This means .4 of one unit in the number 267. What its place value is in the final result depends upon the characteristic. The digits of the result are 2674.

The characteristic 1 shows that the desired number is greater than the first power of 10, but less than the second power of 10 or 100. Hence, the decimal point must be placed between 6 and 7, and the final result is 26.74.

NOTES: 1. For reasons similar to those of Note 1, p. 86, the correction should be carried to one place only.

2. At most the following should be written:

$$\begin{array}{rrl} \log n = & 1.4271 & \qquad \text{tab. diff. } 16 \\ \text{mantissa for } 267 = & .4265 & \qquad \tfrac{6}{16} = .4 \\ \hline \text{diff.} & 6 & \end{array}$$

Therefore, $n = 26.74.$

WRITTEN EXERCISES

Find the number whose logarithm is:

1. 0.7305.	4. 2.9023.	7. $\bar{1}.1962$.	10. $\bar{3}.9485$.
2. 0.5029.	5. 3.1467.	8. $\bar{2}.0342$.	11. 4.6987.
3. 1.4682.	6. 3.6020.	9. $\bar{3}.3920$.	12. $\bar{2}.6376$.

SUMMARY

I. **Definitions.**

1. Exponents indicating the powers of a *base* and used for the purposes of calculation are called *logarithms*. Sec. 126.

2. The integral part of a logarithm is called its *characteristic*, and the decimal part its *mantissa*. Sec. 128.

II. **Laws.**

1. $10^m \cdot 10^r = 10^{m+r}$. The logarithm of a product is the sum of the logarithms of the factors.

2. $\dfrac{10^m}{10^r} = 10^{m-r}$. The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.

3. $(10^m)^r = 10^{mr}$. The logarithm of a number with an exponent is the product of the exponent and the logarithm of the number. Sec. 134.

REVIEW

WRITTEN EXERCISES

Find exactly or approximately by use of logarithms the value of:

1. $\sqrt{2}$.	4. $\sqrt[5]{7}$.	7. $\sqrt[3]{756}$.	10. $(1.03)^7$.
2. $\sqrt{5}$.	5. $\sqrt{926}$.	8. $\sqrt[5]{812}$.	11. $(1.04)^{10}$.
3. $\sqrt[3]{9}$.	6. $\sqrt{656}$.	9. $(1.5)^4$.	12. $(1.06)^9$.

13. $\dfrac{(164)(798)}{779}$.

14. $\dfrac{(732)(774)}{(731)(671)}$.

15. $\sqrt{(624)(598)(178)}$.

16. $\sqrt{\dfrac{(651)(654)(558)}{763}}$.

17. It is known that the volume of a sphere is $\frac{4}{3}\pi r^3$, r being the length of the radius. Using 3.14 as the approximate value of π, find by logarithms the volume of a sphere of radius 7.3 in.

18. Find, as above, the volume of a sphere whose radius is 36.4 ft.

Calculate by logarithms:

1. $\dfrac{(132)(1837)}{167}$.

2. $\dfrac{(2076)(379)}{173}$.

3. $\dfrac{(3059)(349)}{(19)(23)(2443)}$.

4. $\sqrt{\dfrac{(294)(1842)}{307}}$.

5. $\sqrt{\dfrac{(2373)(675)}{113}}$.

6. $\sqrt[3]{.0578}$.

7. $\dfrac{(15.61)^2}{(700)^{\frac{1}{3}}}$.

8. $\sqrt{\dfrac{(7688)(7719)}{(248)(249)}}$.

9. $(217.6)(.00681)$.

10. $\dfrac{[\sqrt{278.2}\,(2.578)]^2}{\sqrt[3]{.00231}\cdot\sqrt{76.19}}$.

11. Given $a = 0.4916$, $c = 0.7544$, and $b = c^2 - a^2$. Find b.

SUGGESTION. $b = (c - a)(c + a)$.

12. It is known that in steam engines, the piston head's average velocity (c) per second is approximately given by the formula:

$$c = 1.7\sqrt[3]{s\sqrt{\frac{p}{15}}},$$

where s denotes the distance over which the piston moves (expressed in the same unit as c), and p the number of pounds pressure in the cylinder.

(1) Find c, if $s = 32.5$ in., $p = 110$ lb.

(2) Find p, if $c = 15$ ft., $s = 2.6$ ft.

CHAPTER VI

IMAGINARY AND COMPLEX NUMBERS

137. Imaginary Numbers. The numbers defined in what precedes have all had positive squares. Consequently, among them the equation $x^2 = -3$, which asks, "What is the number whose square is -3?" has no solution.

A solution is provided by defining a new number, $\sqrt{-3}$, as a number whose square is -3. Similarly we define $\sqrt{-a}$, where a denotes a positive number, as a number whose square is $-a$.

The square roots of negative numbers are called **imaginary numbers.**

138. If a is positive, $\sqrt{-a}$ may be expressed $\sqrt{a}\ \sqrt{-1}$.

Similarly,
$$\sqrt{-5} = \sqrt{5(-1)} = \sqrt{5}\ \sqrt{-1}.$$
$$\sqrt{-49} = \sqrt{49(-1)} = 7\sqrt{-1}.$$

139. Real Numbers. In distinction from imaginary numbers, the numbers hitherto studied are called **real numbers.**

WRITTEN EXERCISES

Express as in Sec. 138:

1. $\sqrt{-9}$.	4. $\sqrt{-100}$.	7. $\sqrt{-18}$.	10. $\sqrt{-12}$.
2. $\sqrt{-16}$.	5. $-\sqrt{-64}$.	8. $-\sqrt{-32}$.	11. $\sqrt{-50}$.
3. $\sqrt{-25}$.	6. $\sqrt{-8}$.	9. $-\sqrt{-7}$.	12. $-\sqrt{-75}$.

140. The positive square root of -1 is frequently denoted by the symbol i; that is, $\sqrt{-1} = i$.

Using this we write:

$$\sqrt{-5} = \sqrt{5} \cdot i.$$
$$\sqrt{-49} = \pm 7\,i.$$
$$\sqrt{-75\,a^2b} = \sqrt{3 \cdot 25\,a^2b \cdot -1} = 5\,a\sqrt{3\,b} \cdot i.$$

NOTE. Throughout this chapter the radical sign is taken to mean the positive root only.

WRITTEN EXERCISES

Rewrite the following, using the symbol i as in Sec. 140:

1. $2 + \sqrt{-4}$.
2. $3 - \sqrt{-9}$.
3. $4 + \sqrt{-4}$.
4. $5 - \sqrt{-16}$.
5. $25 - \sqrt{-25}$.
6. $5 - \sqrt{-3}$.
7. $3 + \sqrt{-6}$.
8. $7 + \sqrt{-12}$.
9. $12 - \sqrt{-9}$.
10. $2\sqrt{-100}$.
11. $4\sqrt{-(a+b)}$.
12. $\sqrt{a} + \sqrt{-b^2c^2}$.
13. $-\sqrt{-b^2c}$.
14. $a + \sqrt{-(a^2 + x^2)}$.
15. $x + y - \sqrt{-xy^2}$.
16. $p^2 + \sqrt{-(p+q)^3}$.

141. Complex Numbers. A binomial one of whose terms is real and the other imaginary is called a **complex number.**

The general form of a complex number is $a + bi$, where a and b may be any real numbers.

NOTE. Complex numbers are also simply called **imaginary**, any expression which involves i being called imaginary. Single terms in which i is a factor (those which we have called imaginary above) are often called **pure imaginaries**, while the others are called **complex imaginaries**. Thus, $\sqrt{-2}$, $3\sqrt{-a}$, $5\,i$ are pure imaginaries and $1 - \sqrt{-3}$, $a - \sqrt{-b}$ are complex imaginaries.

ORAL EXERCISES

1. Name the real term and the imaginary term in each exercise of the last set.

2. Name the values of a and b in each exercise.

142 Processes with Imaginary and Complex Numbers. After introducing the symbol i for the imaginary unit $\sqrt{-1}$, the operations with imaginary and complex numbers are performed like the operations with real numbers.

I. *Addition and Subtraction.*

EXAMPLE

Add $\sqrt{-9}$, $-\sqrt{-25}$, $\sqrt{-3}$.

$$\begin{aligned} \sqrt{-9} &= 3i. \\ -\sqrt{-25} &= -5i. \\ \sqrt{-3} &= -\sqrt{3}\cdot i. \end{aligned}$$

$\therefore$ the sum is $(3-5-\sqrt{3})i = -(2+\sqrt{3})i$.

WRITTEN EXERCISES

Add:

1. $2i, 3i, -i$.
2. $\sqrt{16}\,i, -2i$.
3. $\sqrt{-16}, -2\sqrt{-1}$.
4. $\sqrt{-4}, \sqrt{-9}, \sqrt{-1}$.
5. $6-\sqrt{-5}, 2\sqrt{-4}, \sqrt{-25}$.
6. $3+4i, 2-3i, 5+5i$.
7. $\sqrt{-9x^2}, -\sqrt{-8x^2}$.
8. $\sqrt{-(a+b)^2}, -\sqrt{(b+c)^2}$.
9. $2\sqrt{-32a^3}, 3\sqrt{-8a^3}, 6\sqrt{2}i$.
10. $\sqrt{3}\,i-1, \sqrt{2}\,i+2, i-2\sqrt{2}$.

II. *Multiplication.*

To multiply complex numbers we apply the fact that $\sqrt{-1}\cdot\sqrt{-1}=-1$, or $i^2=-1$, since the square of the square root of a number is the number itself.

EXAMPLES

Multiply:

1. $\sqrt{-16}$ by $\sqrt{-9}$.

$$\begin{aligned} \sqrt{-16} &= 4\sqrt{-1} = 4i. \\ \underline{\sqrt{-9}} &\underline{= 3\sqrt{-1} = 3i.} \end{aligned}$$

$\therefore$ the product is $12(\sqrt{-1})^2 = (12)(-1) = -12$.

This may be written $(4i)(3i) = 12i^2 = -12$.

2. $3-\sqrt{-3}$ by $2-\sqrt{-5}$.

$$\begin{array}{l} 3-\sqrt{3}\,i \\ \underline{2-\sqrt{5}\,i} \\ 6-2\sqrt{3}\,i \\ \underline{\quad -3\sqrt{5}\,i+\sqrt{15}\,i^2} \\ 6-(2\sqrt{3}+3\sqrt{5})i-\sqrt{15}. \end{array}$$

3. $a+bi$ by $a-bi$.

$$\begin{array}{lcr} & a+bi & \\ & \underline{a-bi} & \\ a^2+abi & & \\ \underline{\quad -abi} & \underline{-b^2i^2} & \\ a^2 & +b^2. & \end{array}$$

143. $a+bi$ and $a-bi$ are called *conjugate* complex numbers.

WRITTEN EXERCISES

Multiply:

1. $5-3i$ by $5+3i$.
2. $3+\sqrt{-3}$ by $2+\sqrt{-5}$.
3. $5-2\sqrt{-1}$ by $3+2\sqrt{-1}$.
4. $5+\sqrt{-3}$ by $5-\sqrt{-3}$.
5. $3-\sqrt{-2}$ by $3+2\sqrt{-2}$.
6. $1-\sqrt{-7}$ by $2+3\sqrt{-7}$.
7. $4+i$ by $5-i$.
8. $a+xi$ by $a-xi$.
9. a^2+b^2i by a^2-b^2i.
10. $\sqrt{r}+3i$ by $\sqrt{r}-3i$.
11. $\sqrt{-25}$ by $\sqrt{-9}$ by $\sqrt{-5}$.
12. $\sqrt{-a}$ by $\sqrt{-b}$ by $-ci$.

III. *Division.*

Fractions (that is, indicated quotients) may be simplified by rationalizing the denominator (Sec. 94, p. 47).

For example:

1. $$\frac{\sqrt{-7}}{\sqrt{-5}}=\frac{\sqrt{-7}\,\sqrt{-5}}{\sqrt{-5}\,\sqrt{-5}}=\frac{\sqrt{7}\cdot\sqrt{5}\,(-1)^2}{-5}=\frac{\sqrt{35}}{5}.$$

2. $$\frac{2+\sqrt{-3}}{3+\sqrt{-5}}=\frac{(2+\sqrt{-3})(3+\sqrt{-5})}{(3-\sqrt{-5})(3+\sqrt{-5})}=\frac{6+3\sqrt{-3}+2\sqrt{-5}-\sqrt{15}}{9-(-5)}$$
$$=\frac{1}{14}(6+3\sqrt{-3}+2\sqrt{-5}-\sqrt{15}).$$

3. $$\frac{x+yi}{x-yi}=\frac{(x+yi)^2}{(x-yi)(x+yi)}=\frac{x^2+2xyi-y^2}{x^2+y^2}.$$

WRITTEN EXERCISES

Write in fractional form and rationalize the denominators:

1. $\sqrt{-6}\div\sqrt{2}$.
2. $1\div(a+xi)$.
3. $\sqrt{-3}\div\sqrt{-5}$.
4. $\sqrt{ax}\div\sqrt{-a}$.
5. $1\div(2-\sqrt{-3})$.
6. $4\sqrt{-1}\div-2\sqrt{-4}$.
7. $a\div(a-bi)$.
8. $(a+bi)\div(a-bi)$.
9. $(3+6i)\div(5+4i)$.
10. $(\sqrt{3}-9i)\div(\sqrt{2}-9i)$.
11. $(x-\sqrt{-7})\div(x+\sqrt{-7})$.
12. $\left(a-\dfrac{\sqrt{-5}}{2}\right)\div\left(a+\dfrac{\sqrt{-5}}{2}\right)$.
13. $(\sqrt{-2}+\sqrt{-5})\div(\sqrt{-5}-\sqrt{-2})$.

144. Powers of the Imaginary Unit. Beginning with $i^2 = -1$ and multiplying successively by i we find:

$i^2 = -1.$	$i^6 = i^4 \cdot i^2 = i^2 = -1.$
$i^3 = i^2 \cdot i = -i.$	$i^7 = i^6 \cdot i = -1 \cdot i = -i.$
$i^4 = i^2 \cdot i^2 = -1(-1) = +1.$	$i^8 = i^4 \cdot i^4 = (+1)^2 = +1.$
$i^5 = i^4 \cdot i = i.$	$i^9 = i^8 \cdot i = i.$

145. By means of the values of i^2, i^3, i^4, any power of i can be shown to be either $\pm i$ or ± 1.

For example: $i^{63} = i^{60} \cdot i^3 = (i^4)^{15} \cdot i^3 = 1^{15} \cdot i^3 = i^3 = -i.$

WRITTEN EXERCISES

Simplify similarly:

1. i^9.	4. i^{16}.	7. i^{54}.	10. i^{143}.
2. i^{10}.	5. i^{21}.	8. i^{56}.	11. i^{400}.
3. i^{12}.	6. i^{27}.	9. i^{198}.	12. i^{3001}.

Perform the operations indicated:

13. $(1+i)^2$

14. $(1-i)^3$.

15. $(1-i)^3 \cdot i^4$.

16. $\left(\dfrac{-1+3i}{2}\right)^3$.

17. $(1+i) \cdot i^6$.

18. $\left(\dfrac{-1-3i}{2}\right)^3$.

19. $(1+i) \cdot (1-i)^2$.

20. $(1+i)^2 \div (1-i)^2$.

IMAGINARIES AS ROOTS OF EQUATIONS

146. Complex numbers often occur as roots of quadratic equations.

EXAMPLE

Solve: $$x^2 + x + 1 = 0. \quad (1)$$

$$x^2 + x = -1. \quad (2)$$

Completing the square, $$x^2 + x + \tfrac{1}{4} = \tfrac{1}{4} - 1. \quad (3)$$

$$\therefore x + \tfrac{1}{2} = \pm\sqrt{-\tfrac{3}{4}}. \quad (4)$$

$$\therefore x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i. \quad (5)$$

TEST: $(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i)^2 + (-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i) + 1 = 0.$

WRITTEN EXERCISES

Solve and test, expressing the imaginary roots in the form $a + bi$:

1. $x^2 + 5 = 0.$
2. $x^2 + 2x + 2 = 0.$
3. $x^2 - x + 1 = 0.$
4. $x^2 + x + 5 = 0.$
5. $x^2 + 2x + 37 = 0.$
6. $x^2 - 8x + 25 = 0.$
7. $x^2 - 6x + 10 = 0.$
8. $m^2 + 4m + 85 = 0.$
9. $x^2 + 10x + 41 = 0.$
10. $x^2 + 30x + 234 = 0.$
11. $y^2 - 4y + 53 = 0.$
12. $z^2 - 6z + 90 = 0.$
13. $p^2 + 20p + 104 = 0.$
14. $2x^2 + 4x + 3 = 0.$
15. $3x^2 + 2x + 1 = 0.$
16. $12t^2 + 24 = 0.$
17. $6w^2 + 30 = 0.$
18. $8t^2 + t + 6 = 0.$
19. $7x^2 + x + 5 = 0.$
20. $6x^2 + 3x + 1 = 0.$
21. $4x^2 + 4x + 3 = 0.$
22. $12x^2 + x + 1 = 0.$
23. $8v^2 + 3v + 6 = 0.$
24. $w^2 + 5w + 6 = 0.$
25. $9z^2 + 2z + 5 = 0.$
26. $7x^2 - 3x + 4 = 0.$
27. $15z^2 + 5z - 1 = 0.$
28. $16x^2 - 8x + 1 = 0.$
29. $10x^2 - 2x + 3 = 0.$
30. $7t^2 - t + 1 = 0.$

147. The occurrence of imaginary roots in solving equations derived from problems often indicates the impossibility of the given conditions.

EXAMPLE

A rectangular room is twice as long as it is wide; if its length is increased by 20 ft. and its width diminished by 2 ft., its area is doubled. Find its dimensions.

Solution. 1. Let $x =$ the width of the room, and $2x$ its length.

2. Then $(2x + 20)(x - 2) = 2 \cdot 2x \cdot x$, or $x^2 - 8x + 20 = 0$.

3. Solving (2), $x = 4 \pm 2i$.

The fact that the results are complex numbers shows that no actual room can satisfy the conditions of the problem.

WRITTEN EXERCISES

Solve and determine whether or not the problems are possible:

1. In remodeling a house a room 16 ft. square is changed by lengthening one dimension a certain number of feet and by diminishing the other by twice that number. The area of the resultant room is 296 sq. ft.; what are its dimensions?

2. A triangle has an altitude 2 in. greater than its base, and an area of 32 sq. ft.; find the length of its base.

3. A train moving x mi. per hour travels 90 mi. in $15-x$ hours. What is its rate per hour?

SUMMARY

I. **Definitions.**

1. The square roots of negative numbers are called *imaginary numbers.* Sec. 137.

2. In distinction from imaginaries, the rational and irrational numbers hitherto studied are called *real numbers.* Sec. 139.

3. A *complex number* is a binomial, one of whose terms is a real number and the other an imaginary number. Sec. 141.

II. **Processes.**

1. After introducing the symbol i for the imaginary unit $\sqrt{-1}$, the operations with imaginary and complex numbers are performed like the operations with real numbers. Sec. 142.

2. Any power of i can be expressed by $\pm i$ or ± 1. Sec. 145.

3. The solution of quadratic equations may yield complex numbers. In problems this often indicates the impossibility of the given conditions. Secs. 146, 147.

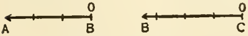
SUPPLEMENTARY WORK

Graphical Representation

We have seen that positive integers and fractions can be represented by lines.

Thus, the line AB represents 3, and the line BC represents $3\frac{1}{2}$.

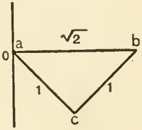
Similarly, we have seen that negative integers and fractions, which for a long time were considered to be meaningless, can be represented by lines.

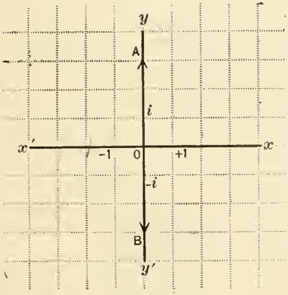


Thus, the line BA represents -3, and the line CB represents $-3\frac{1}{2}$.

Irrational numbers can also be represented by lines.

Thus, in the right-angled triangle abc, the line ab represents the $\sqrt{2}$.





Like the negative number the imaginary number remained uninterpreted several centuries. But this number also can be represented graphically.

Thus, if a unit length on the y-axis be chosen to represent $\sqrt{-1}$ or i, the negative unit $-\sqrt{-1}$ or $-i$ should evidently be laid off in the opposite direction. $3\sqrt{-1}$ or $3i$ would then be represented by OA and $-3i$ by OB, as in the figure, and others similarly.

The reason for placing $\sqrt{-1}$ or i on a line at right angles to the line on which real numbers are plotted may be seen in

the fact that multiplying 1 by $\sqrt{-1}$ twice changes $+1$ into -1. On the graph $+1$ can be changed into -1 by turning it through 180°. If multiplying 1 by $\sqrt{-1}$ twice turns the line 1 through 180°, multiplying 1 by $\sqrt{-1}$ once should turn $+1$ through 90°.

For example:

1. Represent graphically $\sqrt{-4}$:

$\sqrt{-4} = \sqrt{4}\, i = 2\,i$; this is represented by a line 2 spaces long drawn upward on the y-axis.

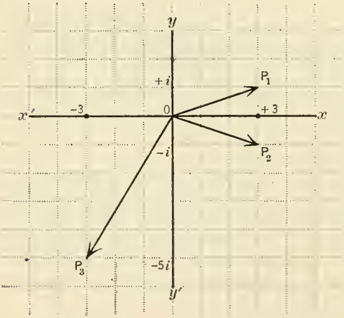
2. Represent graphically $-\sqrt{-3}$:

$-\sqrt{-3} = -\sqrt{3}\,i = -1.7\,i$ (approximately); this is represented by a line 1.7 i spaces long drawn downward on the y-axis.

WRITTEN EXERCISES

Represent graphically:

1. $3\,i$.	5. $-5\,i$.	9. $-5\sqrt{-4}$.
2. $-2\,i$.	6. $5\,i$.	10. $-3\,i$.
3. $\sqrt{-9}$.	7. $\sqrt{-3}$.	11. $+2\sqrt{-3}$.
4. $\sqrt{-16}$.	8. $-\sqrt{-12}$.	12. $5\sqrt{-9}$.



Complex numbers may be represented graphically by a modification of the plan used in representing imaginary numbers.

EXAMPLES

1. Represent graphically $3 + i$.

To do this 3 is laid off on the axis of real numbers, (xx'), and i upward on the axis of imaginaries (yy'). As in other graphical work this locates the point P_1 which is taken to represent the complex number, $3 + i$.

The number $\sqrt{a^2+b^2}$ is called the *modulus* of the complex number $a+bi$. As appears from the figure, $OP_1 = \sqrt{3^2+1^2}$, and hence OP_1 represents the modulus of $3+i$.

2. Represent graphically $3-i$.

The point P_2 is the graph of the complex number $3-i$, and OP_2 represents its modulus.

3. Represent graphically $-3-5i$.

The point P_3 is the graph of the complex number $-3-5i$, and OP_3 represents its modulus.

We have thus *interpreted by means of diagrams* positive and negative integers, positive and negative fractions, positive and negative irrational numbers, and positive and negative complex numbers; in fact, all of the numbers used in elementary algebra.

CHAPTER VII

QUADRATIC EQUATIONS

GENERAL FORM

148. The **general form** for a quadratic polynomial with one unknown quantity is $ax^2 + bx + c$, where a, b, and c denote any algebraic expressions not involving x, and where a is not zero. If a is zero the polynomial is linear.

For example: **1.** $5x^2 - 7x + 8$.

Here $a = 5$, $b = -7$, $c = 8$.

2. $\dfrac{7m}{2n+1}x^2 + 3x - \dfrac{5m}{2n-1}$.

Here $a = \dfrac{7m}{2n+1}$, $b = 3$, $c = \dfrac{-5m}{2n-1}$.

WRITTEN EXERCISES

Put the following expressions into the form $ax^2 + bx + c$:

1. $3x + 5x(x-2) + 4(x^2-5)$.
2. $7(4x-1) + (x+3)(x-2)$.
3. $a(bx+c)(2dx+3e)$.
4. $(x+q) - q(x^2 - 11)$.
5. $(ax+b)(cx+d)$.
6. $(x^2 - a) + (x^2 - b)$.
7. $(x+q)(x+p) - (x-q)(2x-p)$.
8. $\left(\dfrac{x}{2} + \dfrac{1}{3}\right)^2 - \dfrac{2}{3}\left(\dfrac{9x}{8} - \dfrac{16}{3}\right)^2$.
9. $x^2 + ab - ax - b(a + x + x^2)$.
10. $x(x-2)(x-4) - x^2(x-5)$.
11. $(2x+1)^2 - (3x+1)^2 + (4x+1)^2$.
12. $(x-1)(x-2)(x-3) - (x+1)(x+2)(x+3)$.

149. Similarly, every quadratic equation can be put into the form $ax^2+bx+c=0$ by transposing all terms to the left member and then putting the polynomial which constitutes the left member into the form ax^2+bx+c.

EXAMPLES

1. $(3x+5)(2x-7)=3x^2-4$,

then $6x^2-11x-35=3x^2-4$,

or, $3x^2-11x-31=0$.

Here $a=3,\ b=-11,\ c=-31$.

2. $(mx+3a)^2=mx^2-5(amx-2)$,

then $m^2x^2+6amx+9a^2=mx^2-5amx+10$,

or, $(m^2-m)x^2+11amx+9a^2-10=0$.

Here $a=m^2-m,\ b=11am,\ c=9a^2-10$.

WRITTEN EXERCISES

Put the following equations into the form $ax^2+bx+c=0$:

1. $(x-1)^3=(x+1)^3$.

2. $x^2+ex=fx+g(x+e)$.

3. $(7x-1)^2=3x+2$.

4. $\dfrac{(5a-x)^2}{2a-x}=18a-2x$.

5. $\left(\dfrac{x}{a+1}+a\right)^2=\left(\dfrac{x}{a-1}-a\right)^2$.

6. $\dfrac{a-8x}{a+6x}=\dfrac{a-4x}{a-3x}\cdot\dfrac{a-5x}{a+5x}$.

7. $(x+1)(x^2-1)=(x^2+1)(x+2)$.

8. $\left(\dfrac{x}{2a}+\dfrac{3a}{4}\right)^2=\left(\dfrac{x}{2b}-\dfrac{3a}{4}\right)\left(\dfrac{x}{5}+\dfrac{4}{9}\right)$.

9. $\dfrac{2a+5b+3x}{3a-5b+3x}=\dfrac{a+b}{a-b}\cdot\dfrac{3a-b+2x}{2a-b+3x}$.

10. $(2a-4+x)^2+4(a+4+x)^2=(3a+6+2x)^2$.

11. $(2x+4b-3)^2+(2x+2b+11)^2$
$=(x+3b-8)^2+(3x+3b+8)^2$.

METHODS OF SOLUTION

150. General Solution. By solving the general quadratic equation $ax^2+bx+c=0$, general formulas for the roots are obtained.

Solve: $$ax^2+bx+c=0. \quad (1)$$

Dividing by a, which is not 0, $$x^2+\frac{bx}{a}+\frac{c}{a}=0. \quad (2)$$

Adding $\frac{b^2}{4a^2}$ to complete the square and subtracting the same, $$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}=0. \quad (3)$$

$$\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)=0. \quad (4)$$

Writing the second term as the square of its square root, $$\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2=0. \quad (5)$$

Factoring (5),

$$\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)=0. \quad (6)$$

$$\therefore x=-\frac{b}{2a}+\left(\frac{\sqrt{b^2-4ac}}{2a}\right) \text{ and } x=-\frac{b}{2a}-\left(\frac{\sqrt{b^2-4ac}}{2a}\right). \quad (7)$$

Denoting these roots by r_1 and r_2:

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a}.$$

$$r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

By substituting in these formulas the values, including the signs, that a, b, c have in any particular equation, the roots of that equation are obtained. This is called **solution by formula.**

EXAMPLE

Solve: $$3x^2-9x+5=0. \quad (1)$$

Here, $$a=3,\ b=-9,\ c=5, \quad (2)$$

and $$x=\frac{-(-9)\pm\sqrt{(-9)^2-4\cdot 3\cdot 5}}{2\cdot 3}. \quad (3)$$

$$=\frac{9\pm\sqrt{81-60}}{6}=\frac{9\pm\sqrt{21}}{6}. \quad (4)$$

WRITTEN EXERCISES

Solve by formula:

1. $x^2 - x - 1 = 0$.
2. $x^2 + 3x + 1 = 0$.
3. $x^2 + 2x - 1 = 0$.
4. $x^2 - 4x + 4 = 0$.
5. $x^2 - 5x + 6 = 0$.
6. $x^2 - 3x + 2 = 0$.
7. $x^2 - 13x + 9 = 0$.
8. $2x^2 - 7x - 3 = 0$.
9. $x^2 - x + 6 = 0$.
10. $2x^2 - x + 2 = 0$.
11. $3x^2 - 2x + 1 = 0$.
12. $7x^2 + 6x - 4 = 0$.
13. $4x^2 - 12x + 9 = 0$.
14. $3x^2 + 5x - 2 = 0$.
15. $5x^2 - 4x + 6 = 0$.
16. $7x^2 + 5x - 8 = 0$.

151. Literal Quadratic Equations. When any of the coefficients of a quadratic equation involve letters, the equation is called a **literal quadratic equation.**

Such equations are solved in the usual way.

EXAMPLES

1. Solve: $$x^2 + 6mx + 8 = 0. \qquad (1)$$

$$x^2 + 6mx = -8 \qquad (2)$$

Completing the square, $$x^2 + 6mx + 9m^2 = 9m^2 - 8. \qquad (3)$$

$$\therefore (x + 3m)^2 = 9m^2 - 8. \qquad (4)$$

$$\therefore x + 3m = \pm\sqrt{9m^2 - 8}. \qquad (5)$$

$$\therefore x = -3m \pm \sqrt{9m^2 - 8}. \qquad (6)$$

2. Solve: $$t^2 + gt + h = 0. \qquad (1)$$

Here $$a = 1, \quad b = g, \quad c = h. \qquad (2)$$

Hence, by Sec. 150, $$t = \frac{-g \pm \sqrt{g^2 - 4h}}{2}. \qquad (3)$$

3. Solve: $$gt^2 + 2vt = 2s. \qquad (1)$$

$$gt^2 + 2vt - 2s = 0. \qquad (2)$$

Here $$a = g, \quad b = 2v, \quad c = -2s. \qquad (3)$$

Hence, by Sec. 150, $$t = -\frac{2v}{2g} \pm \frac{1}{2g}\sqrt{(2v)^2 - 4g(-2s)}. \qquad (4)$$

$$= -\frac{v}{g} \pm \frac{1}{g}\sqrt{v^2 + 2gs}. \qquad (5)$$

$$= \frac{1}{g}(-v \pm \sqrt{v^2 + 2gs}). \qquad (6)$$

WRITTEN EXERCISES

Solve:

1. $t^2 + at = k.$
2. $u^2 + ku + 1 = 0.$
3. $v^2 + mv = 1.$
4. $ax^2 + bx + c = 0.$
5. $x^2 + ax + b = 0.$
6. $m^2x^2 + 2mx = -1.$
7. $x^2 + 2px - 1 = 0.$
8. $4x^2 - 4ax + 16 = 0.$
9. $a^2x^2 + 2ax + 5 = 0.$
10. $m^2x^2 + 4mx - 6 = 0.$
11. $x^2 - 4ax = 9.$
12. $t^2 - 8t + 24d = 9d^2.$
13. $5ax^2 + 3bx + 2b^3 = 0.$
14. $ay^2 - (a - b)y - b = 0.$
15. $b^2x^2 - 2bx = ac - 1.$
16. $w^2 + 4aw + a^2 = 0.$
17. $x^2 - 3ax + 10a^2 = 0.$
18. $v^2 - 4amv = (a^2 - m^2)^2.$
19. $2x^2 - 3x = a(3 - 4x).$
20. $w^2 - a^2 = 2b(a - w).$

152. Collected Methods. We have used three methods of solving quadratic equations:

1. *Factoring.*

Equation	Factors	Roots
$x^2 - 3x + 2 = 0.$	$(x - 2)(x - 1).$	$x = 2, x = 1.$
$x^2 - (a + b)x + ab = 0.$	$(x - a)(x - b).$	$x = a, x = b.$

2. *Completing the square.*

Equation	Solution	Roots
$x^2 + x + 2 = 0.$	See First Course, Sec. 285, p. 225.	$x = \dfrac{-1 \pm \sqrt{-7}}{2}.$
$ax^2 + bx + c = 0.$	See Sec. 150.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

3. *Formula.*

Equation	Solution	Roots
$3x^2 + 2x - 7 = 0.$	See Sec. 150.	$x = \dfrac{-1 \pm \sqrt{22}}{3}.$
$ax^2 + bx + c = 0.$	See Sec. 150.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

WRITTEN EXERCISES

Solve by factoring:

1. $x^2-x-6=0$.
2. $x^2-x-2=0$.
3. $x^2+x-2=0$.
4. $x^2+x-6=0$.
5. $x^2+3x+2=0$.
6. $x^2-x-30=0$.
7. $x^2+x-12=0$.
8. $x^2-3x+2=0$.
9. $x^2+11x+30=0$.
10. $x^2-7x+12=0$.

Solve by completing the square:

11. $x^2+x+1=0$.
12. $x^2+3x+1=0$.
13. $x^2-\frac{1}{2}x+1=0$.
14. $x^2-.9x+.5=0$.
15. $x^2-5x+10=0$.
16. $x^2-16x+60=0$.
17. $x^2+\frac{3}{4}x+\frac{1}{4}=0$.
18. $x^2+1.5x-3.5=0$.

Solve by formula:

19. $3x^2+x+5=0$.
20. $2x^2-5x-3=0$.
21. $4x^2+3x-1=0$.
22. $5x^2+2x+6=0$.
23. $x^2+x+1=0$.
24. $x^2+1=0$.
25. $x^2+15x+56=0$.
26. $x^2+8x+33=0$.
27. $x^2-10x+34=0$.
28. $2x^2+3x-27=0$.

Solve and test, using whichever of the methods in Sec. 152 seems most convenient:

29. $9y^2-4=0$.
30. $6x^2-13x+6=0$.
31. $5x^2-4x+4=0$.
32. $t^2+11t+30=0$.
33. $6s^2-5s-6=0$.
34. $6r^2-2r-4=0$.
35. $w^2+4w-3=0$.
36. $\dfrac{2x-1}{2x+1}+\dfrac{2x+1}{2x-1}=3$.
37. $x^2-2x+3=0$.
38. $x^2-0.3x+0.9=0$.
39. $x^2-1.1x+1.2=0$.
40. $11x^2+1=4(2-x)^2$.
41. $x^2+(a+b)x+ab=0$.
42. $x^2-(b+c)x+bc=0$.
43. $2ax+(a-2)x-1=0$.
44. $\dfrac{a^2}{b+x}-\dfrac{a^2}{b-x}=c$.

45. $\dfrac{2}{x-4}+\dfrac{3}{x-6}+\dfrac{5}{x-2}=0.$

46. $\frac{1}{2}(x-1)(x-2)=(x-2\frac{2}{3})(x-1\frac{3}{4}).$

47. The product of two consecutive positive integers is 306. Find the integers.

SOLUTION.

1. Let x be the smaller integer.
2. Then $x+1$ is the larger.
3. $\therefore\ x(x+1)$ is their product.
4. $\therefore\ x(x+1)=306$, by the given conditions.
5. $\therefore\ x^2+x-306=0$, from (4).
6. $\therefore\ x=\dfrac{-1\pm\sqrt{1+1224}}{2}=\dfrac{-1\pm 35}{2}=17$, or -18, solving (5).

Since the integers are to be positive, the value -18 is not admissible. $x=17$, $\therefore\ x+1=18$, and the integers are 17 and 18.

TEST. $17\cdot 18=306$.

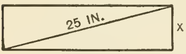
48. There is also a pair of consecutive negative integers whose product is 306. What are they?

49. If the square of a certain number is diminished by the number, the result is 72. Find the number.

50. A certain number plus its reciprocal is -2. What is the number?

51. A certain positive number minus its reciprocal is $\frac{5}{6}$. What is the number? What negative number has the same property?

52. One perpendicular side of a certain right triangle is 31 units longer than the other; the square of their sum exceeds the square of the hypotenuse by 720. Find the sides.



53. The perimeter of the rectangle shown in the figure is 62 in. Find the sides.

54. In a right triangle of area 60 sq. ft. the difference between the perpendicular sides is 7. Find the three sides.

55. The sum of the hypotenuse and one side of a right triangle is 162, and that of the hypotenuse and the other side is 121. What are the sides?

RELATIONS BETWEEN ROOTS AND COEFFICIENTS

153. Relation of Roots to Coefficients. By adding and multiplying the values found for the roots (Sec. 150), we obtain respectively,

$$r_1 + r_2 = -\frac{b}{a}.$$

$$r_1 r_2 = \frac{c}{a}.$$

Applying this result to the equation $x^2 + px + q = 0$, we have:

$$r_1 + r_2 = -p,$$

$$r_1 r_2 = q.$$

In words:

In the equation $x^2 + px + q = 0$, the coefficient of x with its sign changed is the sum of the roots, and the absolute term is their product.

Every quadratic equation can be put into the form $x^2 + px + q = 0$ by dividing both members by the coefficient of x^2.

154. By means of Sec. 153 a quadratic equation may be written whose roots are any two given numbers.

EXAMPLES

1. Write an equation whose roots are $2, -3$.

$$-p = r_1 + r_2 = 2 + (-3) = -1. \quad \therefore p = 1$$

$$q = r_1 r_2 = 2(-3) = -6.$$

$\therefore x^2 + x - 6 = 0$ is the equation sought.

2. Write an equation whose roots are $\frac{1}{2} + \sqrt{-3}, \frac{1}{2} - \sqrt{-3}$.

$$-p = r_1 + r_2 = (\tfrac{1}{2} + \sqrt{-3}) + (\tfrac{1}{2} - \sqrt{-3}) = 1. \quad \therefore p = -1$$

$$q = r_1 r_2 = (\tfrac{1}{2} + \sqrt{-3})(\tfrac{1}{2} - \sqrt{-3}) = \tfrac{1}{4} - (-3) = \tfrac{13}{4}.$$

$\therefore x^2 - x + \frac{13}{4} = 0$ is the equation sought.

WRITTEN EXERCISES

Write the equations whose roots are:

1. $4, 5$.	**3.** $24, 30$.	**5.** $a, -b$.
2. $\frac{3}{4}, \frac{4}{3}$.	**4.** $8\frac{3}{5}, 10$.	**6.** $8, -40$.

7. $7,\ -1\frac{3}{7}$.
8. $-4,\ +4$.
9. $\frac{2}{3} \pm \sqrt{-5}$.
10. $-5,\ -20$.
11. $-\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$.
12. $\frac{5}{6} \pm \frac{1}{6}\sqrt{-47}$.
13. $a - bi,\ a + bi$.
14. $1 + 2i,\ 1 - 2i$.
15. $\frac{1}{2} - \sqrt{2},\ \frac{1}{2} + \sqrt{2}$.

155. Testing Results. The ultimate test of the correctness of a solution is that of substitution; but this is not always convenient, especially when the roots are irrational. In such cases, the relations between the roots and coefficients are of use.

For example: Solving $2x^2 - 5x + 6 = 0$,

or $x^2 - \frac{5}{2}x + 3 = 0$, the roots are

$r_1 = \frac{5}{4} + \frac{1}{4}\sqrt{-23}$ and $r_2 = \frac{5}{4} - \frac{1}{4}\sqrt{-23}$.

Adding, $-(r_1 + r_2) = -\frac{10}{4} = -\frac{5}{2}$, the coefficient of x.

Multiplying, $r_1 r_2 = (\frac{5}{4})^2 - (\frac{1}{4}\sqrt{-23})^2 = \frac{25}{16} + \frac{23}{16} = 3$, the absolute term.

Therefore, the roots are correct. (Sec. 153.)

156. In what follows, the coefficients a, b, c, are restricted to rational numbers.

157. Character of the Roots. By examining the formula for the roots, $-\dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$, it appears that the character of the roots as real or imaginary, rational or irrational, equal or unequal, depends upon the value of the expression $b^2 - 4ac$.

1. If $b^2 - 4ac$ is positive, the roots are real.

Thus, in $x^2 + 4x - 3 = 0$, $b^2 - 4ac = 16 + 12$, or 28, $\therefore$ the roots are real and unequal.

2. If $b^2 - 4ac$ is a perfect square, the indicated square root can be extracted, and the roots are rational.

Thus, in $x^2 - 4x + 3 = 0$, $b^2 - 4ac = 16 - 12$, or 4, $\therefore$ the roots are rational and unequal.

3. If $b^2 - 4ac$ is not a perfect square, the indicated root cannot be extracted and the roots are irrational.

Thus, in $x^2 + 5x + 1 = 0$, $b^2 - 4ac = 25 - 4 = 21$, $\therefore$ the roots are irrational.

4. If $b^2 - 4ac = 0$, the radical is zero, and the two roots are equal.

Thus, $x^2 - 10x + 25 = 0$, $b^2 - 4ac = 100 - 4 \cdot 25 = 0$, $\therefore$ the roots are equal.

5. If $b^2 - 4ac$ is negative, the roots are imaginary.

Thus, in $2x^2 - x + 1 = 0$, $b^2 - 4ac = 1 - 8$, or -7, $\therefore$ the roots are complex numbers.

Consequently, it is merely necessary to calculate $b^2 - 4ac$ to know in advance the nature of the roots of a quadratic equation.

158. Discriminant. Because its value determines the character of the roots, the expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

ORAL EXERCISES

Without solving the equations, find the nature of the roots of:

1. $x^2 + x - 20 = 0.$
2. $x^2 + x - 3 = 0.$
3. $2x^2 - x + 2 = 0.$
4. $3x^2 - x + 3 = 0.$
5. $2x^2 + 2x - 4 = 0.$
6. $5x^2 - 3x + 6 = 0.$
7. $3x^2 - 4x + 5 = 0.$
8. $6x^2 + x - 1 = 0.$
9. $\dfrac{1}{4x} + 1 = x.$
10. $7x^2 + 3x - 4 = 0.$
11. $-4x + 8x^2 + 1 = 0.$
12. $5 + 4x^2 - 3x = 0.$
13. $7x + 6 + x^2 = 0.$
14. $-6x + 9x^2 + 3 = 0.$
15. $x^2 - 6x + 4 = 0.$
16. $5x^2 - \frac{1}{2} + x = 0.$

159. The relation $x^2 + px + q = x^2 - (r_1 + r_2)x + r_1 r_2$ may be written:

$$(1) \qquad x^2 + px + q = (x - r_1)(x - r_2).$$

And since $x^2 + px + q = \dfrac{ax^2 + bx + c}{a}$ in which $p = \dfrac{b}{a}$, $q = \dfrac{c}{a}$, we have

$$(2) \qquad ax^2 + bx + c = a(x - r_1)(x - r_2).$$

160. The solution of a quadratic equation, therefore, enables us to factor every polynomial of either form (1) or (2).

Since r_1 and r_2 involve radicals:

1. *The factors will generally be irrational.*

2. *The factors will be rational when* r_1 *and* r_2 *are so; that is, when* $b^2 - 4ac$ *is a perfect square.*

3. *The two factors involving* x *will be the same when the roots are equal; that is, when* $b^2 - 4ac = 0$.

In the last case the expressions are squares and

(1) becomes $(x - r_1)^2$, and

(2) becomes $[\sqrt{a}(x - r_1)]^2$.

EXAMPLES

	Trinomial	$b^2 - 4ac$	Nature of Factors
1.	$3x^2 - 7x + 2$	$49 - 4 \cdot 3 \cdot 2 = 25$	rational of 1st degree.
2.	$3x^2 - 7x + 3$	$49 - 4 \cdot 3 \cdot 3 = 13$	irrational.
3.	$2x^2 - 8x + 8$	$64 - 4 \cdot 2 \cdot 8 = 0$	equal.

ORAL EXERCISES

By means of the above test, select the squares; also the trinomials with rational factors of the 1st degree:

1. $8x^2 - 8x + 2.$
2. $\frac{y^2}{3} + 4y + 12.$
3. $3x^2 + 3x + 1.$
4. $3z^2 + 2z + 12.$
5. $x^2 + 3x - 2.$
6. $a^2x^2 + 2ax + 1.$
7. $4x^2 + 4x + 1.$
8. $x^2 - 8x + 15.$
9. $6x^2 + 5x - 4.$
10. $6x^2 - 5x + 9.$
11. $4x^2 - 4x - 3.$
12. $8x^2 - 9x + 3.$

161. The actual factors of any quadratic trinomial of the form $ax^2 + bx + c$ can be found by solving the quadratic equation:

$$ax^2 + bx + c = 0,$$

and substituting the roots in the relation:

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

EXAMPLE

Factor: $6x^2 + 5x - 4.$ *(1)*

Solving $6x^2 + 5x - 4 = 0$, $x = -\frac{4}{3}$, *(2)*

$x = \frac{1}{2}.$

Substituting $-\frac{4}{3}$ for r_1, $\frac{1}{2}$ for r_2 and 6 for a. $a(x - r_1)(x - r_2) = 6(x + \frac{4}{3})(x - \frac{1}{2}).$ *(3)*

Therefore, $6x^2 + 5x - 4 = 6(x + \frac{4}{3})(x - \frac{1}{2}).$ *(4)*

WRITTEN EXERCISES

Factor:

1. $3x^2-2x-5$.	5. $10w^2-12w+2$.	9. $6x^2-7x+3$.
2. $9x^2-3x-6$.	6. $9v^2-17v-2$.	10. $5x^2-40x+6$.
3. $6y^2+y-1$.	7. $6x^2+25x+14$.	11. $a^{2m}-2a^m-3$.
4. $15y^2-4y-35$.	8. $2z^2+5z+2$.	12. c^4-13c^2+36.

GRAPHICAL WORK

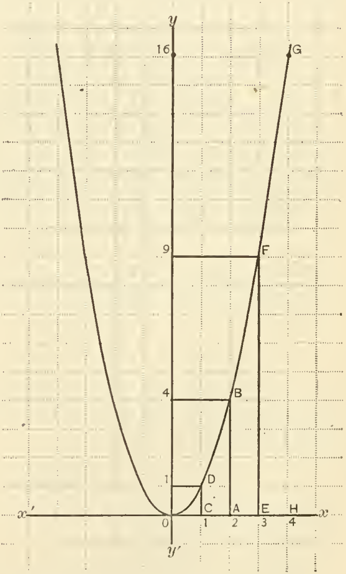
162. PREPARATORY.

1. By counting spaces read the length of EF in the figure.

2. Is it the square of the length of OE?

3. Answer similar questions for GH and OH.

Every point of the curve is so located that *the length of its ordinate is the square of its abscissa.*



163. Quadratic expressions may be represented graphically.

For example:

The curve in the figure is the graph of $y = x^2$. That is, the length of CD is the square of that of OC; the length of AB is the square of that of OA; etc.

WRITTEN EXERCISES

1. Construct on a large sheet of squared paper the points corresponding to this table of squares.

NUMBER	SQUARE
-5	25
-4	16
-3	9
-2	4
-1	1
-0	0
1	1
2	4
3	9
4	16
5	25

2. Then sketch a smooth curve through the points beginning with -5, 25.

The work should be carefully done, and the result preserved for later use. As there are no negative values of x^2, the x-axis should be taken near the lower edge of the paper. The unit should be chosen quite large; for example, 10 spaces. Then the table might include squares of numbers increasing by tenths: 1, 1.1, 1.3, etc. The curve will be a graphical table of squares and square roots.

3. Read to one decimal place from the graph $\sqrt{2}$; $\sqrt{3}$; $\sqrt{5}$; $\sqrt{6}$; $\sqrt{7}$; $\sqrt{8}$

Every y-distance is the square of the corresponding x-distance; and every x-distance is the square root of the corresponding y-distance. We see that for every y-distance there are two corresponding x-distances, one plus and the other minus, corresponding to the two square roots. Thus the points of the curve for which $y = 4$ are those whose values of x are 2 and -2 respectively, *i.e.* $\sqrt{4} = \pm 2$.

164. Graphical Solution of Quadratic Equations. Any value of x which satisfies the system

$$\begin{cases} y = x^2, \\ y = -px - q \end{cases}$$

makes x^2 equal to $-px - q$, or $x^2 + px + q = 0$.

The values of x satisfying the system may be read from the graph of $y = x^2$.

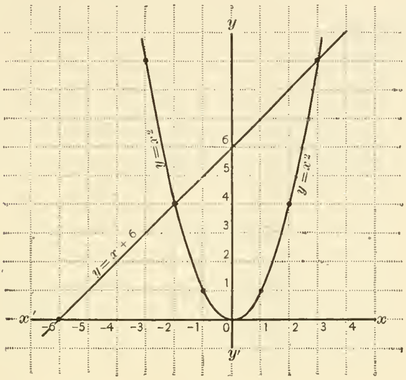
EXAMPLE

Solve graphically $x^2 - x - 6 = 0$.

1. Construct the graph of $y = x^2$. (As in Sec. 163.)
2. Construct the graph of $y = x + 6$.
3. They intersect at points for which $x = -2$ and $+3$.

$\therefore$ the roots of $x^2 - x - 6 = 0$ are -2, 3.

NOTES. 1. Step 2 may be done by simply noting two points of the graph of $y = x + 6$ and laying a ruler connecting them. The roots can



be read while the ruler is in position, and thus the same graph for $y = x^2$ can be used for several solutions.

2. The equation must first be put in the form $x^2 + px + q = 0$, if not so given.

WRITTEN EXERCISES

Solve graphically:

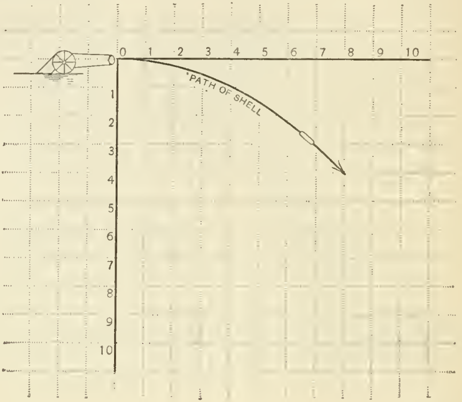
1. $x^2 - 5x + 6 = 0$.
2. $x^2 + 3x + 2 = 0$.
3. $x^2 - 2x - 3 = 0$.
4. $x^2 + 2x - 3 = 0$.
5. $x^2 - 3x - 40 = 0$.
6. $x^2 + 4x + 4 = 0$.
7. $2x^2 - x - 1 = 0$.
8. $3x^2 - 2x - 1 = 0$.
9. $x^2 + x + \frac{1}{4} = 0$.
10. $x^2 + x - 2 = 0$.
11. $4x^2 + 4x + 1 = 0$.
12. $x^2 - 9 = 0$.

13. The path of a projectile fired horizontally from an elevation, as at 0 in the figure on p. 114, with a given velocity may

be represented by the graph of the equation $y = \frac{gx^2}{2v^2}$, where $g = 32$ and v is the initial velocity of the projectile in feet per second. Let $v = 16$ ft. per second and compute the numbers to complete the table of values of x and y.

TABLE

x	y
0	0
1	( )
4	( )
8	( )
9	( )
16	( )
24	( )
32	( )
48	( )



Read from the graph of this table the horizontal distance traveled by the projectile when it is 4 ft. below the starting point.

14. Construct similarly the path of a projectile whose initial velocity is 32 ft. per second.

15. A cannon of a fort on a hill is 300 ft. above the plane of its base. The cannon can be charged so as to give the projectile an initial velocity of 100 ft. per second. What range does the cannon cover?

16. The enemy is observed at a point known to be $2\frac{1}{2}$ mi. from the foot of the vertical line in which the cannon stands. With what initial velocity must the ball be discharged to strike the enemy?

CERTAIN HIGHER EQUATIONS SOLVED BY THE AID OF QUADRATIC EQUATIONS

165. We have found the general solution of linear and quadratic equations with one unknown. Equations of the third and the fourth degree can also be solved generally by algebra, and certain types of equations of still higher degree as well; but these solutions do not belong to an elementary course. We shall take up only certain equations of higher degree whose solution is readily reduced to that of quadratic equations.

EXAMPLES

1. Solve: $x^4 - 9x^2 + 8 = 0.$ (*1*)

Let $y = x^2$; then the given equation becomes, $y^2 - 9y + 8 = 0.$ (*2*)

Solving for y, $y = 8$ or $1.$ (*3*)

Therefore $x^2 = 8.$ (*4*)

Or, $x^2 = 1.$ (*5*)

Solving (*4*), (*5*), $x = \pm\sqrt{8},\ \pm 1.$ (*6*)

The four values of x are the four roots of the given equation of the fourth degree. Test them all by substitution.

2. Solve: $x^6 - 3x^3 - 4 = 0.$ (*1*)

Let $y = x^3$, then $y^2 - 3y - 4 = 0.$ (*2*)

Solving (*2*), $y = 4,$

and $y = -1.$ (*3*)

$\therefore$ by the substitution in (*2*), $x^3 = 4$, or $x^3 - 4 = 0,$

and $x^3 = -1$, or $x^3 + 1 = 0.$ (*4*)

Factoring (*4*), $x^3 - 4 = (x - \sqrt[3]{4})(x^2 + \sqrt[3]{4}\cdot x + \sqrt[3]{4^2}) = 0.$ (*5*)

and $x^3 + 1 = (x + 1)(x^2 - x + 1) = 0.$ (*6*)

Solving (*5*), $x = \sqrt[3]{4}$, or $\sqrt[3]{4}(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})$, or $\sqrt[3]{4}(-\frac{1}{2} - \frac{1}{2}\sqrt{-3}).$ (*7*)

Solving (*6*), $x = -1$, or $\frac{1}{2} - \frac{1}{2}\sqrt{-3}$, or $\frac{1}{2} + \frac{1}{2}\sqrt{-3}).$ (*8*)

3. Solve: $(x^2 - 3x + 1)(x^2 - 3x + 2) = 12.$ (*1*)

This may be written $(x^2 - 3x + 1)[(x^2 - 3x + 1) + 1] = 12.$ (*2*)

If y is put for $x^2 - 3x + 1$, the equation becomes $y(y + 1) = 12,$ (*3*)

or $y^2 + y - 12 = 0.$ (*4*)

Solving, $y = 3 \text{ or } -4.$ (5)

Then from (3), $x^2 - 3x + 1 = 3,$ (6)

and $x^2 - 3x + 1 = -4.$ (7)

Solving (6), $x = \dfrac{3 \pm \sqrt{17}}{2}.$ (8)

Solving (7), $x = \dfrac{3 \pm \sqrt{-11}}{2}.$ (9)

These are the four roots of the given equation of the fourth degree.

WRITTEN EXERCISES

Solve as above:

1. $x^6 - 7x^3 + 6 = 0.$
2. $x^8 - 3x^4 + 2 = 0.$
3. $x^{10} - 5x^5 + 6 = 0.$
4. $x^4 + 13x^2 + 36 = 0.$
5. $x^4 - 3x^2 + 1 = 0.$
6. $12 - x^4 = 11x^2.$
7. $ax^{2n} - bx^n + c = 0.$
8. $\dfrac{1}{x^2+1} + \dfrac{1}{x^2+2} = \dfrac{1}{x^2+3}.$
9. $x^3 = 1 - x^6.$
10. $x^2 + 5 = \dfrac{5}{x^2+3}.$
11. $x^{2n} - 4x^n - 5 = 0.$
12. $2x^6 + 5x^3 + 2 = 0.$
13. $x^4 + ax^2 - 8a^2 = 0.$
14. $(x^2+4)^2 - 4(x^2+4) + 4 = 0.$
15. $x^2 + 3x = 1 - \dfrac{1}{x^2+3x+1}.$
16. $(x^2 - 3x + 1)(x^2 - 3x + 2) = 12.$
17. $(x^2 - 1)^2 + 2(x^2 - 1) + 1 = 0.$
18. $(x^2 + 5x - 1)(x^2 + 5x + 1) = -1.$

166. Binomial Equations. Equations of the form $x^n \pm a = 0$ are called **binomial equations.** The simpler cases admit of being solved by elementary processes.

EXAMPLES

1. Solve: $x^3 - 1 = 0, \text{ or } x^3 = 1.$ (1)

Factoring $x^3 - 1$, $(x-1)(x^2+x+1) = 0.$ (2)

Finding equations equivalent to (2), $x = 1,\ x^2 + x + 1 = 0.$ (3)

Solving (3), $x = 1,\ x = \dfrac{-1 - \sqrt{-3}}{2}.$ (4)

Thus we have found the three numbers such that the cube of each is 1, or *the three cube roots of unity.*

Verify this statement by cubing each number in step (4).

Note that $\left(-\frac{1}{2}+\frac{1}{2}\sqrt{-3}\right)^2=\frac{-1-\sqrt{-3}}{2}=\frac{-1-i\sqrt{3}}{2}$,

also that $\left(-\frac{1}{2}-\frac{1}{2}\sqrt{-3}\right)^2=\frac{-1+\sqrt{-3}}{2}=\frac{-1+i\sqrt{3}}{2}$.

Hence, if ω stands for one of the complex cube roots of unity, ω^2 is the other.

Every number has three cube roots; for example, the cube roots of 8 are 2, and $2\,\omega$ and $2\,\omega^2$.

Verify this by cubing 2, $2\,\omega$, and $2\,\omega^2$.

2. Solve: $x^4+1=0$, or $x^4=-1$. *(1)*

Factoring x^4+1, $(x^2-i)(x^2+i)=0$. *(2)*

Solving *(2)*, $x^2=i$, $x^2=-i$. *(3)*

Solving *(3)*, $x=\pm\sqrt{i}$, $x=\pm\sqrt{-i}$. *(4)*

These are the four numbers, each of which raised to the fourth power equals -1, or *the four fourth roots of* -1.

WRITTEN EXERCISES

1. Find the 3 cube roots of -1 by solving $x^3+1=0$.
2. Find the 4 fourth roots of 1 by solving $x^4-1=0$.
3. Find the 6 sixth roots of 1 by solving
$$(x^6-1)=(x^3-1)(x^3+1)=0.$$
4. Find the 4 fourth roots of 16 by solving $x^4-16=0$.
5. Find the 3 cube roots of 8 by solving $x^3-8=0$.
6. Show that the square of either irrational cube root of -1 is the negative of the other irrational cube root.
7. Show that the sum of the three cube roots of unity is zero; also that the sum of the six sixth roots of unity is zero.

SUMMARY

I. Forms and Definitions.

1. A general form of the quadratic equation is:
$$ax^2+bx+c=0. \qquad \text{Sec. 148.}$$
2. The general forms of the roots of this equation are:
$$r_1=\frac{-b+\sqrt{b^2-4\,ac}}{2\,a}.$$
$$r_2=\frac{-b-\sqrt{b^2-4\,ac}}{2\,a}. \qquad \text{Sec. 150.}$$

3. The *discriminant* of the quadratic equation is $b^2 - 4ac$.

If $b^2 - 4ac$ is greater than 0, the roots are real and unequal.
If $b^2 - 4ac$ is equal to 0, the roots are real and equal.
If $b^2 - 4ac$ is less than 0, the roots are complex numbers.
If $b^2 - 4ac$ is a perfect square, the roots are rational.
If $b^2 - 4ac$ is not a perfect square, the roots are irrational.

Secs. 157, 158.

4. The discriminant of the quadratic equation enables us to find by inspection the nature of the factors of a quadratic expression. Sec. 160.

5. In the quadratic equation $x^2 + px + q = 0$, the coefficient of x with its sign changed is the sum of the roots, and the absolute term is their product. Sec. 153.

6. Equations of the form $x^n \pm a = 0$ are binomial equations. Sec. 166.

7. The three cube roots of unity are 1, ω, and ω^2, where $\omega = \dfrac{-1 + i\sqrt{3}}{2}$ or $\dfrac{-1 - i\sqrt{3}}{2}$. Sec. 166.

II. **Processes.**

1. *Methods of solution:* By factoring, by completing the square, by formula. Sec. 152.

2. Certain higher equations may be solved by the methods of quadratic equations. Sec. 165.

REVIEW

WRITTEN EXERCISES

Solve:

1. $x^2 = 6x - 5.$
2. $w^2 - w - 1 = 0.$
3. $x^2 - 6x - 7 = 0.$
4. $v^2 + 2v + 6 = 0.$
5. $x^2 - 5x + 1 = 0.$
6. $2x^2 - x + 3 = 0.$
7. $3x^2 - x + 7 = 0.$
8. $x^2 - 5x + 11 = 0.$
9. $x^2 - 14x + 5 = 0.$
10. $3x^2 - 9x - \frac{5}{2} = 0.$
11. $x^2 - 4x - 9021 = 0.$
12. $x^2 - 4x + 9021 = 0.$
13. $x^2 + 4x - 9021 = 0.$
14. $x^2 + 4x + 9021 = 0.$
15. $x^2 + 30x + 221 = 0.$
16. $x^2 - 30x - 221 = 0.$

17. $x + \frac{a^2}{x} = \frac{a^2}{b} + b.$

18. $cx^2 + bx + a = 0.$

19. $(x-2)(x+3) = 16.$

20. $x^2 = 6x + 16.$

21. $24 - 10x = x^2.$

22. $(x-1)^2 = x + 2.$

23. $5x + x^2 + 6 = 0.$

24. $x^2 - 9x + 14 = 0.$

25. $x^2 + 3x - 70 = 0.$

26. $4x^2 - 4x - 3 = 0.$

27. $3x^2 - 7x + 2 = 0.$

28. $x^2 - 10x + 21 = 0.$

29. $x^2 - 10x + 24 = 0.$

30. $x^2 + 10x + 24 = 0.$

31. $x^2 = 9x^2 - (x+1)^2.$

32. $9x^2 + 4x - 93 = 0.$

33. $4x^2 + 3x - 22 = 0.$

34. $6x^2 - 13x + 6 = 0.$

35. $(x+1)(x+2) = x + 3.$

36. $\frac{a^2}{b+x} + \frac{a^2}{b-x} = c.$

37. $\frac{a(a^2+x^2)}{a+x} = ax + b^2.$

38. $\frac{x+1\frac{1}{3}}{x+2} - \frac{x+12}{2(x+19)} = \frac{1}{2}.$

39. $5x - \frac{3(x-1)}{x-3} = 2x + \frac{3(x-2)}{2}.$

40. $\frac{x-a}{x-b} - \frac{x-b}{x-a} = \frac{(a-b)^2}{(x-a)(x-b)}.$

41. $(x-1)^2(x+3) = x(x+5)(x-2).$

42. $x^2 - 6acx + a^2(9c^2 - 4b^2) = 0.$

State what can be known by means of the discriminant and without solving, concerning the factors of the following trinomials:

43. $3x^2 - 2x + 1.$

44. $4x^2 + 11x - 1.$

45. $5y^2 + 20y + 20.$

46. $2x^2 - x + 3.$

Similarly, what can be known about the roots of:

47. $t^2 - 9 = 0$?

48. $5x^2 + x + 2 = 0$?

49. $z^2 - z - 1 = 0$?

50. $5x^2 = 9x$?

How must a be chosen in order that:

51. The roots of $x^2 + ax + 5 = 0$ shall be imaginary?

52. The roots of $ax^2 + 6x + 1 = 0$ shall be real?

53. The roots of $x^2 + 4x + 2a = 0$ shall be real and of opposite signs?

54. The roots of $(a+1)x^2 + 3x - 2 = 0$ shall be imaginary?

55. The roots of $4x^2 - ax + 2 = 0$ shall be real and both positive?

Find the values of m for which the roots of the following equations are equal to each other. What are the corresponding values of x?

56. $x^2 - 12x + 3m = 0$.

57. $mx^2 + 8x + m = 0$.

58. $4x^2 + mx + x + 1 = 0$.

59. $mx^2 + 3mx - 5 = 0$.

60. A number increased by 30 is 12 less than its square. Find the number.

61. The product of two consecutive odd numbers is 99. What are the numbers? Is there more than one set?

62. Find two consecutive even numbers the sum of whose squares is 164.

63. Find a positive fraction such that its square added to the fraction itself makes $\frac{4}{9}$.

64. If a denotes the area of a rectangle and p its perimeter, show that the lengths of the sides are the roots of the equation

$$x^2 - \frac{p}{2}x + a = 0.$$

65. The diagonal and the longer side of a rectangle are together equal to 5 times the shorter side, and the longer side exceeds the shorter by 35 meters. Find the area of the rectangle.

66. A rug 9 ft. by 12 ft. covers $\frac{1}{2}$ the floor of a room, and can be laid so that the uncovered strip of floor about the rug is of the same breadth throughout. Find the dimensions of the room.

67. A company of soldiers attempts to form in a solid square, and 56 are left over. They attempt to form in a square with 3 more on each side, and there are 25 too few. How many soldiers are there?

68. It took a number of men as many days to dig a ditch as

there were men. If there had been 6 more men the work would have been done in 8 days. How many men were there?

69. Solve: $$2x^2 + 6x + c = 0.$$
What value has c if the two values of x be equal? If they be reciprocal?

70. A public library spends \$180 monthly for books. In June the average cost per book was 15¢ less than in May, and 60 books more were bought. How many books were bought in May?

71. When water flows from an orifice in a tank the square of the velocity (v) equals $2g$ times the height (h) of the surface above the orifice. Write the equation that denotes this fact. g is the "constant of gravity" and may be taken as 32.

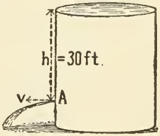


Fig. 1.

72. What does the square of the velocity, (v^2), at A in Fig. 1 equal? Find this velocity.

73. What would be the velocity of the water if an opening were made halfway up the tank shown in the figure?

74. Find the velocity with which water rushes through an opening at the base of a dam against which the water stands 25 ft. high.

75. The distance on the level from the bottom of the vertical wall to the point where the stream reaches the ground is called the range. The range in Fig. 2 is a. If t is the number of seconds taken by the water to reach the ground after leaving o, then $a = vt$. Take a to be 10 ft., and find t after computing v as above.

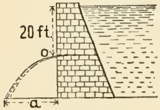


Fig. 2.

76. Water flowing from a leak in a dam reaches the plane of the base 20 ft. from the dam. The opening is 10 ft. below the surface of the water. How long after leaving the dam does the water strike the base?

77. Water flowing from an orifice in a standpipe reaches the level ground 40 ft. from the base in 3 seconds. How high is the column of water above the orifice?

SUPPLEMENTARY WORK

ADDITIONAL EXERCISES

Solve for x:

1. $\dfrac{1}{a}+\dfrac{1}{a+x}+\dfrac{1}{a^2-x^2}=0.$

2. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}-\dfrac{1}{a+b+x}=0.$

3. $(2a-5-x)^2+(3a-3x)^2=(a+5-2x)^2.$

4. $(3x-4a+3b)^2+(2x+a)^2=(x-4a+b)^2+(2x-3a+4b)^2.$

5. $(7a+3b+x)^2+(4a-b-8x)^2-(4a+3b+4x)^2=(7a+b-5x)^2.$

6. $(2a+4b+6x)(3a-9b+x)=(2a-5b+5x)^2.$

7. $(x+a)(5x-3a-4b)=(x+a-2b)^2.$

8. $(5x+4a+3b)(10x-6a+8b)=(5x+a+7b)^2.$

9. $(26x+a+22b)(14x+13a-2b)=(16x+11a+8b)^2.$

10. $(8c+10+4x)(18c+160+24x)=(12c+40+11x)^2.$

11. $\dfrac{14y^2+16}{21}-\dfrac{2y^2+8}{8y^2-11}=\dfrac{2y^2}{3}.$

12. $\dfrac{a^2+x^2}{a^2}=\dfrac{(b+c)^2}{(b-c)^2}.$

13. $\dfrac{3x^2-27}{x^2+3}+\dfrac{90+4x^2}{x^2+9}=7.$

14. $\dfrac{x^2+x-2}{x^2+2x-3}-\dfrac{x+1}{x-3}=0.$

15. $\dfrac{1}{a+b-x}=\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{x}.$

16. $(a-x)\left(1-\dfrac{3a+3x}{c-\frac{1}{2}}\right)-2=\dfrac{1-2a}{1-2c}(c-3)-\dfrac{1+a}{1-\dfrac{1}{2c}}.$

Quadratic Equations involving Radicals

If the equation involves but one radical, the method of Sec. 96, p. 48, can be used.

When the equation resulting from squaring is not directly solvable, substitution is sometimes useful.

EXAMPLE

Solve: $$5x^2-3x+\sqrt{5x^2-3x+2}=18. \qquad (1)$$

Putting $5x^2-3x=y$, the given equation becomes $$y+\sqrt{y+2}=18. \qquad (2)$$

Subtracting y, $$\sqrt{y+2}=18-y. \qquad (3)$$

Squaring, $$y+2=324-36y+y^2. \qquad (4)$$

Rearranging, $$y^2-37y+322=0. \qquad (5)$$

Solving, $$y=\frac{37\pm\sqrt{37^2-4\cdot 322}}{2} \qquad (6)$$

$$=23 \text{ or } 14. \qquad (7)$$

Hence the values of x are determined from $$5x^2-3x=23, \qquad (8)$$

and $$5x^2-3x=14. \qquad (9)$$

Solving these, $$x=\frac{3\pm\sqrt{469}}{10}, \qquad (10)$$

and $$x=2, \text{ or } -1.4. \qquad (11)$$

Test. Substituting, it appears that 2 and -1.4 satisfy the equation. The values $\frac{3\pm\sqrt{469}}{10}$ satisfy

$$5x^2-3x-\sqrt{5x^2-3x+2}=18.$$

Sometimes successive squaring is necessary.

EXAMPLE

Solve: $$\sqrt{2x+6}+\sqrt{3x+1}=8. \qquad (1)$$

Rearranging, $$\sqrt{2x+6}=8-\sqrt{3x+1}. \qquad (2)$$

Squaring, $$2x+6=64-16\sqrt{3x+1}+3x+1. \qquad (3)$$

Collecting, $$16\sqrt{3x+1}=x+59. \qquad (4)$$

Squaring again and collecting, $$x^2-650x+3225=0. \qquad (5)$$

Solving, $$x=5, \text{ or } 645. \qquad (6)$$

Test. Trial shows that the first of these values satisfies the given equation; and it is obvious on inspection that the second cannot satisfy the equation.

Sometimes it is best first to transform the given expression.

EXAMPLE

Solve: $$2x+\sqrt{4x^2+9}=-\frac{2x+1+\sqrt{5x+6}}{2x-\sqrt{4x^2+9}}. \qquad (1)$$

Clearing of fractions, $9=2x+1+\sqrt{5x+6}.$ (2)

Rearranging, $8-2x=\sqrt{5x+6}.$ (3)

Squaring and collecting, $4x^2-37x+58=0.$ (4)

Hence, $x=2$, or $7\frac{1}{4}$. (5)

TEST. By trial, 2 is seen to satisfy the given equation. To avoid the complete work of substituting $7\frac{1}{4}$, we note that every root of (1) must satisfy (3). If $x=7\frac{1}{4}$, the left member of (3) is negative and the right member positive. Hence $7\frac{1}{4}$ is not a root. It would satisfy (3) if the radical had the negative sign.

WRITTEN EXERCISES

Solve:

1. $x-\sqrt{3x+10}=6.$
2. $6x-x^{\frac{1}{2}}-12=0.$
3. $\sqrt{x}-\sqrt{x-15}=1.$
4. $x^{\frac{2}{5}}-x^{\frac{1}{5}}-6=0.$
5. $x^2+x+\sqrt{x^2+x+1}=-1.$
6. $x^2+a+\sqrt{x^2+2a}=-a.$
7. $x^2+6x-9=\sqrt{x^2+6x+7}.$
8. $4\sqrt{x^2+7x+10}=\sqrt{5}\cdot\sqrt{x^2+5x+10}.$
9. $\sqrt{x^2+6x-16}+(x+3)^2=25.$
10. $2x-5x^{\frac{1}{2}}+2=0.$
11. $x^{-\frac{2}{5}}-x^{-\frac{1}{5}}-6=0.$
12. $3x^{-\frac{3}{2}}-4x^{-\frac{3}{4}}=7.$
13. $\sqrt{x^2-5}+\frac{6}{\sqrt{x^2-5}}=5.$
14. $x+2+(x+2)^{\frac{1}{2}}=20.$
15. $x^2+3x-3+\sqrt{x^2+3x+17}=0.$
16. $\sqrt{x+7}+\sqrt{3x-2}=\frac{4x+9}{\sqrt{3x-2}}.$
17. $x+\sqrt{x^2-8}=\frac{3x-\sqrt{x^2-8}}{x-\sqrt{x^2-8}}.$
18. $\sqrt{x(a+x)}+\sqrt{x(a-x)}=2\sqrt{ax}.$

Factoring applied to solving Equations

Factor Theorem. The relation

$$x^2 + px + q = (x - r_1)(x - r_2)$$

is a special case of an important general theorem, known as the **factor theorem**:

If any polynomial in x *assumes the value zero when* a *is substituted for* x, *then* x − a *is a factor of the polynomial.*

We prove this first for the particular polynomial

$$x^4 - 3x^3 + 7x^2 - 2x - 16.$$

By substituting 2 for x we find that the polynomial assumes the value zero.

Suppose the polynomial to be divided by $x - 2$, and denote the quotient by Q and the remainder by R, the latter being numerical.

Then $$x^4 - 3x^3 + 7x^2 - 2x - 16 = (x - 2)Q + R.$$

In this equation substitute 2 for x; the left number becomes zero as just seen; $2 - 2$ is also 0, and 0 times Q is 0, no matter what value Q may have; hence the result of substitution is,

$$0 = 0 + R, \text{ or } R = 0.$$

Consequently $x^4 - 3x^3 + 7x^2 - 2x - 16 = (x - 2)Q$, or $x - 2$ is a factor of $x^4 - 3x^3 + 7x^2 - 2x - 16$. This may now be verified by actual division, but the value of the theorem lies in the fact that by it we know *without actual division* that $x - 2$ is a factor of the polynomial.

Notation for Polynomials. It is convenient to have a symbol to represent any polynomial involving x. For this purpose the symbol $P(x)$ is used. It is read "P of x." $P(2)$ means the *same* polynomial when 2 is substituted for x, $P(-\sqrt{5})$ means the polynomial when $-\sqrt{5}$ is put for x; and $P(a)$ means the polynomial when a is put for x.

General Proof: Let $P(x)$ denote the given polynomial.

Suppose $P(x)$ to be divided by $x - a$. There will be a certain quotient, call it $Q(x)$, and a remainder, R. This remainder will not involve x, otherwise the division could be continued.

We have then: $$P(x) = (x - a)Q(x) + R. \qquad (1)$$

In this equation, put a in place of x:

$$P(a) = (a - a)Q(a) + R. \qquad (2)$$

By hypothesis, $P(a) = 0$, $a - a = 0$, and R remains unchanged when x is replaced by a, for there is no x in R. Hence

$$0 = 0 + R, \text{ or } R = 0. \tag{3}$$

Substituting this value of R in (1),

$$P(x) = (x - a)\,Q(x). \tag{4}$$

That is, $x - a$ is a factor of $P(x)$, as was to be proved.

WRITTEN EXERCISES

In each polynomial substitute the values given for x, and use the factor theorem, when applicable, to determine a factor of the polynomial:

	Polynomial	Values for x
1.	$x^4 + 3x^3 + 4x^2 - 12x - 32$	$1, 2, -2.$
2.	$x^4 - 7x^3 + 14x^2 + x - 21$	$1, -1, 2, 3.$
3.	$2x^4 + 5x^3 - 41x^2 - 64x + 80$	$4, -4, 5, -5.$

By use of the factor theorem, prove that each polynomial has the factor named:

	Polynomial	Factor
4.	$x^3 + 12x^2 + 31x - 20$	$x + 5.$
5.	$\dfrac{x^3 - 1}{x - 1} - \dfrac{a^3 - 1}{a - 1}$	$x - a.$
6.	$x^3 - 2x^2 + \dfrac{3x}{m} - \left(m^3 - 2m^2 + \dfrac{3m}{x}\right)$	$x - m.$
7.	$(x - a)^2 + (x - b)^2 - (a - b)^2$	$x - b.$
8.	$x^3 + 2x^2 + 3x + 2$	$x + 1.$
9.	$x^{3n} + 2x^{2n} + 3x^n + 2$	$x^n + 1.$
10.	$x^3 + ax^2 - a^2x^2 - a^5$	$x - a^2.$
11.	$9x^5 + (3a^2 - 12a)x^4 - 4a^3x^3 + 3a^2x + a^4$	$x + \dfrac{a^2}{3}.$

Roots and Factors. The factor theorem may be worded thus:

If a *is a root of the equation* $P(x) = 0$, *then* $x - a$ *is a factor of the polynomial* $P(x)$.

It is also evident that:

If x − a *is a factor of* P(x), *then* a *is a root of* P(x) = 0.

For, let $\qquad P(x) = (x - a)\,Q(x)$.

Putting a for x, $\qquad P(a) = (a - a)\,Q(a)$.
$\qquad = 0.$

Thus there is a complete correspondence between *root of equation* and *linear factor of polynomial.* As in quadratic polynomials, so also in all higher polynomials: To every linear factor there corresponds a root, and *vice versa.* If we know a linear factor, we can at once determine a root; if we know a root, we can at once write a linear factor.

This property enables us to write out any equation of which all the roots are known.

EXAMPLE

Write the equation whose roots are 2, 3, −1, 0.

The factors are $x - 2$, $x - 3$, $x - (-1)$, and, $x - 0$.
The polynomial is $(x - 2)(x - 3)(x + 1)x$.
The equation is $x^4 - 4x^3 + x^2 + 6x = 0$.

WRITTEN EXERCISES

Write the equations whose roots are:

1. $3, -4, 0$.
2. $2, -2, \sqrt{5}, -\sqrt{5}$.
3. $4, 1 + \sqrt{2}, 1 - \sqrt{2}$.
4. $6, -1, i, -i$.
5. a, b, c.
6. $a, -b, c$.
7. $a, -a, b, -b, 0$.
8. $2 - \sqrt{3}\,i, 2 + \sqrt{3}\,i, 4$.

When a root is known, we thereby know a factor of the given polynomial, and this may aid in finding the other roots of the equation.

EXAMPLE

Find the roots of the equation $x^3 - 3x^2 - 4x + 12 = 0$.

The equation $\qquad x^3 - 3x^2 - 4x + 12 = 0 \qquad (1)$

has the root −2. (Verify by substitution.) Hence the polynomial has the factor $x - (-2)$ or $x + 2$. Finding the other factor by division, the equation may be written,

$$(x + 2)(x^2 - 5x + 6) = 0. \qquad (2)$$

The left member will be zero, either if

$$x + 2 = 0, \tag{3}$$

or

$$x^2 - 5x + 6 = 0. \tag{4}$$

Equation (*3*) has the root -2, and equation (*4*) has the roots 2 and 3. Hence the roots of (*1*) are 2, -2, and 3.

WRITTEN EXERCISES

Each of the following equations has one of the six numbers, ± 1, ± 2, ± 3, as a root. Find one root by trial, and then solve completely as in the example:

1. $x^3 + 5x^2 - 4x - 20 = 0.$
2. $x^3 + 3x^2 - 16x - 48 = 0.$
3. $x^3 - 3x^2 + 5x - 15 = 0.$
4. $x^3 + 2x^2 - ax - 2a = 0.$
5. $x^3 + x^2 + x + 1 = 0.$
6. $x^3 + 3x^2 - 5x - 10 = 0.$
7. $2x^3 - 2x^2 - 17x + 15 = 0.$
8. $x^3 + 3x^2 - 4x - 12 = 0.$
9. $x^3 - (2b + 3)x^2 + (b^2 + 6b - c)x + 3b^2 - 3c^2 = 0.$

10. By finding the h. c. f. of their first members, find the two roots that are common to the equations

$$x^4 - 4x^3 + 2x^2 + x + 6 = 0 \text{ and } 2x^3 - 9x^2 + 7x + 6 = 0.$$

Whenever an equation has the right member zero, any factors that may be given are aids to the solution. The roots corresponding to factors of the first degree can at once be written out. Factors of higher degree determine equations of corresponding degree to be solved.

WRITTEN EXERCISES

Solve by equating each factor to zero:

1. $x(x - 1)(x + 2) = 0.$
2. $(x + 2)(x - 5)(x + \sqrt{3}) = 0.$
3. $(x^2 - 1)(x - 7)(x + 5) = 0.$
4. $x(x + 2i)(x - 7\sqrt{5}) = 0.$
5. $(x - 4)(x^2 - 4)(x + 4) = 0.$
6. $x(x - ai)(x^2 - b^2) = 0.$
7. $x(x - 7)(x + \sqrt{7})(x^2 + 7) = 0.$
8. $(x^2 + 5x + 6)(x^2 - 9x + 14) = 0.$
9. $(x^2 - 7x + 11)(x^2 + 2x + 5) = 0.$

10. $x(x^2-4)(x^2+7x-3)=0.$

11. $(x^2-25)(x^2+3)(x^2-6x+13)=0.$

12. $(x-a)(x+b)x(x^2-2cx+c^2)=0.$

13. $(x^2+2ax+b)(x^2-a^2b)(5x-1)=0.$

Graphical Work

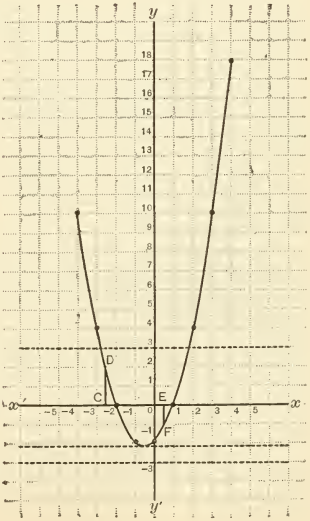
Graphical Representation of Solutions. The solutions of $ax^2+bx+c=0$ may be represented graphically by plotting the curve corresponding to $y=ax^2+bx+c$. The solutions of the equation will be represented by the point where the curve cuts the x-axis.

EXAMPLE

Represent the solutions of $x^2+x-2=0$.

To draw the curve $y=x^2+x-2$, we make a table of a sufficient number of pairs of values of x and y, plot the corresponding points, and sketch a smooth curve through them.

x	y
-4	10
-3	4
-2	0
-1	-2
$-\frac{3}{4}$	$-2\frac{3}{16}$
$-\frac{1}{2}$	$-2\frac{1}{4}$
$-\frac{1}{4}$	$-2\frac{3}{16}$
0	-2
1	0
2	4
3	10
4	18



NOTE. When the integral values do not satisfactorily outline the shape of the curve, fractional values of x must also be used as above.

This graph not only represents the solutions of $x^2+x-2=0$; but, what is far more important, it represents the values and variation of the polynomial x^2+x-2 for all values of x (within the limits of the figure). Thus EF represents the value of the trinomial for $x=\frac{1}{2}$, and CD that for $x=-2\frac{1}{2}$. The figure also tells us that x^2+x-2 is positive when x is negative and numerically greater than -2 (algebraically less than -2). That when x is algebraically less than -2 and increasing, the trinomial is positive and decreasing, reaching the value zero when $x=-2$. As x increases, the trinomial continues to decrease, becoming negative, reaching its least value for $x=-\frac{1}{2}$. It then begins to increase, through negative values, reaching zero when $x=1$. As x increases further, the trinomial continues to increase, becoming positive and continuing to increase as long as x increases.

The graph enables us to read off approximately not only the roots of $x^2+x-2=0$, but also those of any equation of the form $x^2+x-2=a$, if they are real.

For example:

1. The roots of $x^2+x-2=3$ are indicated by the points where the line drawn parallel to the x-axis through the point 3 on the y-axis cuts the graph.

2. The line parallel to the x-axis through the point -3 does not cut the graph at all. This tells us that there are no real values of x which make the trinomial x^2+x-2 equal to -3. The equation $x^2+x-2=-3$ has imaginary roots.

The graph shows (1) that the equation $x^2+x-2=a$ has two real and distinct roots whenever a is positive; also when a ranges from zero to a little beyond -2; (2) that it has imaginary roots when a is less than a certain value between -2 and -3; (3) at a certain point the line parallel to the x-axis just touches the graph. This corresponds to the case of equal roots of the equation. The value of a in this case may be read from the graph as $-2\frac{1}{4}$. This means that the equation $x^2+x-2=-2\frac{1}{4}$, or $x^2+x+\frac{1}{4}=0$, has equal roots. This may be verified by solving the equation or by applying the discriminant (Sec. 384, p. 349).

Note. Since even the best of drawing is not mathematically accurate, results read off from a graph are usually only approximately correct. The closeness of the approximation depends on the degree of accuracy in the drawing.

WRITTEN EXERCISES

Treat each trinomial of Nos. 1 to 9 below as follows:

(1) Draw the graph of the trinomial.

(2) From the graph discuss the variations of the trinomial.

(3) From the graph read off approximately the roots of the equation resulting from equating the trinomial to zero.

(4) If the trinomial be equated to a, read from the graph the range of values of a, for which the roots of the equation are, (i) real and distinct, (ii) real and equal, (iii) imaginary.

(5) Last of all, verify those of the preceding results which relate to roots of the equations by solving the equations or by use of the discriminant.

1. x^2+5x-6. **4.** x^2+4x-5. **7.** x^2+2x+3.

2. x^2+3x+2. **5.** $3x^2-2x$. **8.** x^2-2x-1.

3. $2x^2-5x+2$. **6.** x^2-4. **9.** $8+2x-x^2$.

10. Draw the graph of the function $2x^2+5x-3$, and state how it varies as x varies from a negative value, numerically large at will, through zero to a large positive value. For what values of x is the function positive? For what values negative?

11. What is the graphic condition that ax^2+bx+c shall have the same sign for all values of x? What must therefore be the character of the roots of $ax^2+bx+c=0$, if the trinomial ax^2+bx+c has the same sign for all values of x?

Determine m so that each of the following trinomials shall be positive for all values of x:

12. x^2+mx+5. **13.** $3x^2-5x+m$. **14.** mx^2+6x+8.

CHAPTER VIII

SYSTEMS OF QUADRATIC AND HIGHER EQUATIONS

SIMULTANEOUS QUADRATIC EQUATIONS

167. Two simultaneous quadratic equations with two unknowns cannot in general be solved by the methods used in solving quadratic equations, because an equation of higher degree usually results from eliminating one of the unknowns. But many simultaneous quadratic equations can be solved by quadratic methods, and certain classes of these will be taken up in this chapter.

168. Class I.

A system such that substituting from one equation into the other produces a quadratic equation.

1. *A system of equations composed of a linear equation and a quadratic equation can always be solved by substitution.*

EXAMPLE

Solve: $$\begin{cases} 3x + 4y = 24. & (1) \\ x^2 + y^2 = 25. & (2) \end{cases}$$

From (*1*), $$3x = 24 - 4y. \quad (3)$$

From (*3*), $$x = 8 - \frac{4y}{3}. \quad (4)$$

Substituting (*4*) in (*2*), $$\left(8 - \frac{4y}{3}\right)^2 + y^2 = 25. \quad (5)$$

Simplifying (*5*), $$25y^2 - 192y + 351 = 0. \quad (6)$$

Factoring (*6*), $$(y-3)(25y-117) = 0. \quad (7)$$

Solving (*7*), $$y = 3 \text{ and } y = \frac{117}{25}. \quad (8)$$

From (*4*), $$x = 4 \text{ and } x = \frac{44}{25}. \quad (9)$$

TEST. $3 \cdot 4 + 4 \cdot 3 = 24$ and $\frac{3 \cdot 44}{25} + \frac{4 \cdot 117}{25} = 24.$

$$3^2 + 4^2 = 25.$$

$$\left(\frac{44}{25}\right)^2 + \left(\frac{117}{25}\right)^2 = 25.$$

It should be noted that to every x there corresponds only one particular value of y, and *vice versa*. The proper correspondence may be seen by noticing which value of one unknown furnishes a given value of the other in the process of solution. In the example above, $y = 3$ produces $x = 4$.

2. *When both equations are quadratic, substitution is applicable if the result of substitution is an equation having the quadratic form.*

EXAMPLE

Solve: $\begin{cases} x^2 + y^2 = 25. & (1) \\ xy = 12. & (2) \end{cases}$

From (2),	$x = \frac{12}{y}.$	(3)
Substituting from (3) in (1),	$\left(\frac{12}{y}\right)^2 + y^2 = 25.$	(4)
Simplifying (4), we obtain an equation in quadratic form,	$y^4 - 25y^2 + 144 = 0.$	(5)
Factoring (5),	$(y^2 - 16)(y^2 - 9) = 0.$	(6)
Solving (6),	$y = \pm 4$, and ± 3.	(7)
Substituting (7) in (3),	$x = \pm 3$, and ± 4.	(8)

TEST. Taking both values to be positive, or both to be negative,

$$(\pm 3)^2 + (\pm 4)^2 = 25.$$

$$(\pm 3) \cdot \pm 4 = 12.$$

There are frequently various methods of solving the same problem. Thus, in the last example, multiply (2) by 2, add it to or subtract it from (1); the resulting equations $(x + y)^2 = 49$, and $(x - y)^2 = 9$, can be solved by extracting the square roots, and adding and subtracting the results.

WRITTEN EXERCISES

Solve and test:

1. $x^2 + y^2 = 13,$
   $2x + 3y = 13.$

2. $2x^2 - y^2 = 14,$
   $3x + y = 11.$

3. $3x^2 - 2xy = 15,$
   $2x + 3y = 12.$

4. $x^2 - xy + y^2 = 3,$
   $2x + 3y = 8.$

5. $x^2+y^2=2\,xy,$
$x+y=8.$

6. $x^2+y^2=25,$
$x+y=1.$

7. $x-y=\frac{11}{2},$
$xy=20.$

8. $x^2-y^2=16,$
$x-y=2.$

9. $\frac{x^2}{y^2}=\frac{85}{9}-\frac{4\,x}{y},$
$x-y=2.$

10. $2\,x+y=7,$
$x^2+2\,y^2=22.$

11. $x-y=1,$
$x^2-y^2=16.$

12. $3\,xy+2\,x+y=485,$
$3\,x-2\,y=0.$

13. $x+y=a,$
$x^2+y^2=b.$

14. $x-y=b,$
$xy=a^2.$

15. $2\,x=x^2-y^2,$
$2\,x=4\,xy.$

16. $\frac{1}{y}=\frac{4}{5\,x},$
$x^2-y^2=81.$

169. Class II.

A system in which one equation has only terms of the second degree in x *and* y *can be solved by finding* x *in terms of* y, *or* vice versa, *from this equation and substituting in the other.*

EXAMPLE

Solve: $\begin{cases} x^2-5\,xy+6\,y^2=0. & (1)\\ x^2-y^2=27. & (2)\end{cases}$

Dividing (1) by y^2, $\frac{x^2}{y^2}-\frac{5\,xy}{y^2}+\frac{6\,y^2}{y^2}=0.$ (3)

Simplifying (3), $\left(\frac{x}{y}\right)^2-5\left(\frac{x}{y}\right)+6=0.$ (4)

Factoring (4), $\left(\frac{x}{y}-2\right)\left(\frac{x}{y}-3\right)=0.$ (5)

Solving (5), $\frac{x}{y}=2,\ \frac{x}{y}=3.$ (6)

From (6), $x=2\,y,\ x=3\,y.$ (7)

Substituting $x=2\,y$ in (2), $(2\,y)^2-y^2=27.$ (8)

Solving (8), $y=\pm 3.$ (9)

Substituting (*9*) in $x = 2y$, $$x = \pm 6. \qquad (10)$$

Substituting $x = 3y$ in (*2*), $$(3y)^2 - y^2 = 27. \qquad (11)$$

Solving (*11*), $$y = \frac{\pm 3\sqrt{6}}{4}. \qquad (12)$$

Substituting (*12*) in $x = 3y$, $$x = \frac{\pm 9\sqrt{6}}{4}. \qquad (13)$$

TEST.

Taking both values to be positive or both negative,
$$\begin{cases} (\pm 6)^2 - 5(\pm 3)(\pm 6) + 6(\pm 3)^2 = 0. \\ (\pm 6)^2 - (\pm 3)^2 = 27. \end{cases}$$

Taking the signs as before,
$$\begin{cases} \left(\frac{\pm 9\sqrt{6}}{4}\right)^2 - 5\left(\frac{\pm 9\sqrt{6}}{4}\right)\left(\frac{\pm x}{4\sqrt{6}}\right) + 6\left(\frac{\pm 3}{4\sqrt{6}}\right)^2 = 0. \\ \left(\frac{\pm 9\sqrt{6}}{4}\right)^2 - \left(\frac{\pm 3}{4\sqrt{6}}\right)^2 = 27. \end{cases}$$

NOTES. 1. When the equation in $\frac{x}{y}$, as in the fourth step, cannot be factored by inspection, the formula for solving the quadratic equation is used. Putting $\frac{x}{y} = z$, a quadratic equation is obtained whose solution will lead to the results of step (*7*) above.

2. When an equation has the right member zero, it is unnecessary to divide by x^2 or y^2, if the left member can be factored by inspection. Thus, in (*1*) above, $(x - 3y)(x - 2y) = 0$, hence $x = 3y$ and $x = 2y$, as in (*7*).

WRITTEN EXERCISES

Solve:

1. $x^2 - 2xy - 3y^2 = 0,$
   $x^2 + 2y^2 = 12.$

2. $x^2 + xy + y^2 = 0,$
   $x^2 + y^2 = -\frac{1}{4}.$

3. $6x^2 + 5xy + y^2 = 0,$
   $y^2 - x - y = 32.$

4. $2x^2 - 5xy + 2y^2 = 0,$
   $4x^2 - 4y^2 + 3x = 330.$

5. $x(x + y) = 0,$
   $x^2 - xy + y^2 = 27.$

6. $x^2 - y^2 = 0,$
   $x^2 - 3xy + y = 105.$

7. $3x^2 - 2xy - y^2 = 0,$
   $x + y + y^2 = 32.$

8. $4x^2 + 4xy + y^2 = 0,$
   $x + 3y^2 - 2x = 195.$

9. $x^2 + 3x - 4y + xy = 33,$
   $x^2 + 7xy + 10y^2 = 0.$

10. $x^2 + a^2y^2 = 0,$
    $x^2 + y - y^2 = -a^2.$

170. Class III.

A system of two simultaneous quadratic equations whose terms are of the second degree in x *and* y *with the exception of the absolute terms can be solved by reducing the system to one of Class II.*

This can be done in two ways:

1. *Make their absolute terms alike, and subtract. The resulting equation has every term of the second degree in* x *and* y.

EXAMPLE

Solve: $\begin{cases} x^2 + xy = 66, & (1) \\ x^2 - y^2 = 11. & (2) \end{cases}$

Multiplying (2) by 6, $6x^2 - 6y^2 = 66.$ (3)

Subtracting (1) from (3), $5x^2 - xy - 6y^2 = 0.$ (4)

Factoring (4), $(5x - 6y)(x + y) = 0.$ (5)

Expressing x in terms of y, $x = \frac{6}{5}y$ and $x = -y.$ (6)

Substituting $x = \frac{6}{5}y$ in (2), $\frac{36}{25}y^2 - y^2 = 11.$ (7)

Solving (7), $y = \pm 5.$ (8)

Substituting $y = \pm 5$ in $x = \frac{6}{5}y$, $x = \pm 6.$ (9)

Similarly, substituting $x = -y$ in (2), $y^2 - y^2 = 11.$ (10)

But this leads to $0 = 11.$ (11)

(11) being impossible, the solution is

$x = \pm 6,\ y = \pm 5.$ (12)

TEST. Taking the values to be both positive or both negative, $\begin{cases} (\pm 6)^2 + (\pm 6)(\pm 5) = 66. \\ (\pm 6)^2 - (\pm 5)^2 = 11. \end{cases}$

2. *Substitute* vx *for* y *throughout the equations and solve for* v.

EXAMPLE

Solve: $\begin{cases} 2x^2 - 3xy + y^2 = 4, & (1) \\ x^2 - 2xy + 3y^2 = 9. & (2) \end{cases}$

Putting $y = vx$ in (1) and (2), $2x^2 - 3vx^2 + v^2x^2 = 4,$ (3)

and, $x^2 - 2vx^2 + 3v^2x^2 = 9.$ (4)

Factoring (3) and (4), $x^2(2 - 3v + v^2) = 4.$ (5)

$x^2(1 - 2v + 3v^2) = 9.$ (6)

Equating the values of x^2 in (5) and (6), $\quad \dfrac{4}{2-3v+v^2} = \dfrac{9}{1-2v+3v^2}$. (7)

Clearing (7) of fractions, $\quad 3v^2 + 19v - 14 = 0.$ (8)

Solving (8), $\quad v = -7, \frac{2}{3}.$ (9)

Since $y = vx$, $\quad y = -7x,$ (10)

and, $\quad y = \dfrac{2x}{3}.$ (11)

From (1), $\quad 2x^2 + 21x^2 + 49x^2 = 72x^2 = 4.$ (12)

Solving (12), $\quad x = \pm \dfrac{1\sqrt{2}}{6}.$ (13)

From (10), $\quad y = \mp \dfrac{7\sqrt{2}}{6}.$ (14)

Similarly, from (11) and (1), $\quad x = \pm 3,$ (15)

From (15) and (11), $\quad y = \pm 2.$ (16)

TEST as usual.

WRITTEN EXERCISES

Solve:

1. $x^2 + y^2 = 41,$
   $xy = 20.$

2. $x^2 + xy = a^2,$
   $y^2 + xy = b^2.$

3. $x^2 + xy + y^2 = 7,$
   $3x^2 - xy = 1.$

4. $x^2 + xy - 2y^2 = 4,$
   $x^2 - 3xy + 2 = 0.$

5. $x^2 + 6xy + 2y^2 = 133,$
   $x^2 - y^2 = 16.$

6. $x^2 - 12xy + 119 = 0,$
   $y^2 - 2xy + 24 = 0.$

7. $a^2x^2 + b^2y^2 = c^2,$
   $x^2 - y^2 = c^2d^2.$

8. $x^2 + y^2 = 13,$
   $xy = 6.$

9. $x^2 - xy = 54,$
   $xy - y^2 = 18.$

10. $x^2 + xy = 12,$
    $y^2 + xy = 24.$

11. $4x^2 + 3y^2 = 43,$
    $3x^2 - y^2 = 3.$

12. $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{5}{4},$
    $x^2 + y^2 = 20.$

13. $2x^2 + 3xy + y^2 = 80,$
    $xy - x^2 = 6.$

14. $x^2 - 4y^2 = 20,$
    $xy = 12.$

15. $x^2 - xy + y^2 = 21,$
    $2xy - y^2 = 15.$

16. $4x^2 + 3y^2 = 43,$
    $3x^2 - y^2 - 3 = 0.$

171. Class IV.

A system in which each equation is unaltered when x *and* y *are interchanged can be solved by letting* $x = u + v$ *and* $y = u - v$.

EXAMPLE

Solve: $\begin{cases} x^2 + y^2 - x - y = 78, & (1) \\ xy + x + y = 39. & (2) \end{cases}$

Let $x = u + v$, $y = u - v$, then (*1*) becomes

$$(u + v)^2 + (u - v)^2 - (u + v) - (u - v) = 78, \qquad (3)$$

and (*2*) becomes

$$(u + v)(u - v) + (u + v) + (u - v) = 39. \qquad (4)$$

Simplifying (*3*), $2u^2 + 2v^2 - 2u = 78.$ (*5*)

Simplifying (*4*), $u^2 - v^2 + 2u = 39.$ (*6*)

Dividing (*5*) by 2, and subtracting (*6*) from the result, $2v^2 - 3u = 0.$ (*7*)

Expressing u in terms of v, $u = \frac{2v^2}{3}.$ (*8*)

Substituting (*8*) in (*6*), $\frac{4v^4}{9} + \frac{v^2}{3} = 39.$ (*9*)

Simplifying (*9*), $4v^4 + 3v^2 - 9 \cdot 39 = 0.$ (*10*)

Solving (*10*), $v^2 = -\frac{39}{4},$ (*11*)

and, $v^2 = 9.$ (*12*)

Solving (*11*) and (*12*), $v = \pm \frac{1}{2}\sqrt{-39}.$ (*13*)

Substituting (*13*) and (*14*) in (*8*), $v = \pm 3,$ (*14*)

Then, $u = -\frac{13}{2}$, or 6. (*15*)

Also $x = u + v = \frac{-13 \pm \sqrt{-39}}{2}.$ (*16*)

$= 6 \pm 3 = 9$ or 3, (*17*)

and, $y = u - v = \frac{-13 \mp \sqrt{-39}}{2}.$ (*18*)

Also $= 6 \mp 3 = 3$ or 9. (*19*)

Such equations may often be solved by some of the previous methods, in which case it is usually preferable to do so.

WRITTEN EXERCISES

Solve:

1. $xy-(x+y)-1=0,$
$xy=2.$

2. $\frac{x}{y}+\frac{y}{x}=\frac{5}{2},$
$\frac{1}{x}+\frac{1}{y}=4.$

3. $x^2+y^2=39,$
$y-x=3.$

4. $x^2-xy+y^2-7=0,$
$x-y+1=0.$

5. $x^2-xy+y^2=12,$
$x^2+xy+y^2=4.$

6. $xy=3(x+y),$
$x^2+y^2=160.$

7. $x^2+y^2+x+y=188,$
$xy=77.$

8. $x^2-xy+y^2=19,$
$xy=15.$

9. $2x^2+2y^2-(x-y)=9,$
$y^2=1. \ =1.$

172. The foregoing classes of simultaneous quadratic equations are applied in the following problems.

WRITTEN EXERCISES

1. Two square floors are paved with stones 1 ft. square; the length of the side of one floor is 12 ft. more than that of the other, and the number of stones in the two floors is 2120. Find the length of the side of each floor.

SOLUTION.

Let x be the length in feet of a side of the smaller floor and y be the length of that of the other, then

$$x = y - 12. \qquad (1)$$

and by the given conditions, $$x^2 + y^2 = 2120. \qquad (2)$$

Substituting (1) in (2), $$(y-12)^2 + y^2 = 2120. \qquad (3)$$

Simplifying (3), $$y^2 - 12y - 988 = 0. \qquad (4)$$

Solving (4), $$y = 38 \text{ and } -26, \qquad (5)$$

Substituting (5) in (1), $$x = 26 \text{ and } -38. \qquad (6)$$

The negative values not being admissible, the squares are 26 ft. and 38 ft. on a side. (7)

TEST. $$26^2 + 38^2 = 2120.$$

2. The sum of the sides of two squares is 7 and the sum of their areas is 25. Find the side of each square.

3. The hypotenuse of a certain right triangle is 50, and the length of one of its sides is $\frac{3}{4}$ that of the other. Find the sides of the triangle.

4. The difference between the hypotenuse of a right triangle and the other two sides is 3 and 6 respectively. Find the sides.

5. A number consists of two digits; the sum of their squares is 41. If each digit is multiplied by 5, the sum of these products is equal to the number. Find it.

6. The difference between two numbers is 5; their product exceeds their sum by 13. Find the numbers.

7. In going 120 yd. the front wheel of a wagon makes 6 revolutions more than the rear wheel; but if the circumference of each wheel were increased 3 ft., the front wheel would make only 4 revolutions more than the rear wheel in going the same distance. Find the circumference of each wheel.

8. The diagonal of a rectangle is 13 in.; the difference between its sides is 7 in. Find the sides.

9. The diagonal of a rectangle is 29 yd., and the sum of its sides is 41 yd. Find the sides.

10. The sum of the perimeters of two squares is 104 ft.; the sum of their areas is 346 sq. ft. Find their sides.

11. The difference between the areas of two squares is 231 sq. in.; the difference between their perimeters is 28 in. Find their sides.

12. Two steamers set out simultaneously from San Francisco and Honolulu, 2100 mi. apart, and travel toward each other; they meet 840 mi. from Honolulu in $1\frac{1}{2}$ da. less than the difference between their rates in miles per hour. Find the rate of each per hour.

13. Two trains leave New York simultaneously for St. Louis, which is 1170 mi. distant; the one goes 10 mi. per hour faster than the other and arrives $9\frac{3}{4}$ hr. sooner. Find the rate of each train.

SIMULTANEOUS HIGHER EQUATIONS

173. Certain systems containing higher equations can be solved by the methods of this chapter.

EXAMPLES

1. Solve: $\begin{cases} x^3 + y^3 = 18\,xy, & (1) \\ x + y = 12. & (2) \end{cases}$

Put $x = u + v$, and $y = u - v$,

Then, from (1) $\quad (u + v)^3 + (u - v)^3 = 18(u + v)(u - v), \quad (3)$

and from (2), $\quad (u + v) + (u - v) = 2\,u = 12. \quad (4)$

Combining, (4) and (3), $\quad 216 + 18\,v^2 = 9(36 - v^2). \quad (5)$

Solving (5), $\quad v = \pm 2. \quad (6)$

$\therefore \quad x = u + v = 8$ and $4, \quad (7)$

and $\quad y = u - v = 4$ and $8. \quad (8)$

Test as usual.

2. Solve: $\begin{cases} x^4 + y^4 = 706, & (1) \\ x - y = 2. & (2) \end{cases}$

Put $x = u+v$, $y = u-v$,

Then from (1) $\quad (u+v)^4 + (u-v)^4 = 706, \quad (3)$

and from (2), $\quad (u+v) - (u - v) = 2. \quad (4)$

Simplifying (4), $\quad v = 1. \quad (5)$

From (5) and (3), $\quad (u+1)^4 + (u-1)^4 = 706. \quad (6)$

Simplifying (6), $\quad u^4 + 6\,u^2 - 352 = 0. \quad (7)$

Solving (7), $\quad u^2 = -22. \quad (8)$

and $\quad u^2 = 16. \quad (9)$

$\therefore \quad u = \pm\sqrt{-22},\ \pm 4. \quad (10)$

$\therefore \quad x = u+v = \pm\sqrt{-22}+1. \quad (11)$

or $\quad x = \pm 4 + 1 = 5,\ -3. \quad (12)$

and, $\quad y = u-v = \pm\sqrt{-22}-1. \quad (13)$

or, $\quad y = \pm 4 - 1 = -5,\ +3. \quad (14)$

Test as usual.

WRITTEN EXERCISES

Solve:

1. $x^3 + y^3 = 189,$
   $x + y = 9.$

2. $x^3 + y^3 = 72,$
   $x + y = 6.$

3. $x^3 + y^3 = 189,$
   $x^2y + xy^2 = 180.$

4. $x^4 + y^4 = 81,$
   $x + y = 5.$

5. $x^5 + y^5 = 2,$
   $x + y = 2.$

6. $x^4 + y^4 = 10{,}001,$
   $x - y = 9.$

SUMMARY

1. Certain classes of simultaneous quadratic equations can be solved by quadratic methods: Sec. 167.

(1) *Class I.* A system of equations such that substituting from one equation into the other produces an equation of quadratic form. Sec. 168.

(2) *Class II.* A system in which one equation has only terms of the second degree in x and y can be solved by finding x in terms of y, or *vice versa*, from this equation and substituting in the other. Sec. 169.

(3) *Class III.* A system of two simultaneous quadratic equations whose terms are of the second degree in x and y, with the exception of the absolute terms, can be solved by reducing the system to one of Class II.

This can be done in two ways:

Make their absolute terms alike and subtract. The resulting equation has every term of the second degree in x and y.

Substitute vx for y throughout the equations and solve for v. Sec. 170.

(4) *Class IV.* A system in which each equation is unaltered when x and y are interchanged, can be solved by letting $x = u + v$ and $y = u - v$. Sec. 171.

2. Certain simultaneous higher equations can be solved by quadratic methods. Sec. 173.

REVIEW

WRITTEN EXERCISES

Solve:

1. $x+y=7.5,$
$xy=14.$

2. $3x-2y=0,$
$xy=13.5.$

3. $x+y=7,$
$x^2-y^2=21.$

4. $x-y=5,$
$x^2+y^2=37.$

5. $x-y=1,$
$3x^2+y^2=31.$

6. $x-y=5,$
$x^2+2xy+y^2=75.$

7. $x+y=7(x-y),$
$x^2+y^2=225.$

8. $5(x^2-y^2)=4(x^2+y^2),$
$x+y=8.$

9. $x+y=9,$
$\dfrac{xy}{\sqrt{xy}}=\dfrac{10}{\sqrt{5}}.$

10. $\dfrac{x^2y}{x+5y}=4,$
$xy=20.$

11. $x+y=a,$
$x^3+y^3=b^3.$

12. $x-y=a,$
$x^3-y^3=b^3.$

13. $x^3-y^3=665,$
$x-y=5.$

14. $x^2+xy+y^2=19,$
$xy=6.$

15. $(x+2)(x-3)=0,$
$x^2+3xy+y^2=5.$

16. $x^2-y^2=3,$
$x^2+y^2-xy=3.$

17. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{5}{2},$
$x^2+y^2=90.$

18. $x^2+xy+y^2=7,$
$x^4+x^2y^2+y^4=21.$

19. $x^2+y^2-1=2xy,$
$xy(xy+1)=8190.$

20. $x+y=5,$
$\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}.$

21. $x+y=\dfrac{x}{y}=x^2-y^2.$

22. $x^2-y^2=144,$
$x-y=8.$

23. $x^2+xy=\frac{5}{12},$
$xy+y^2=\frac{5}{18}.$

24. $\dfrac{x}{2}-\dfrac{y}{3}=0,$
$x^2+y^2=5(x+y)+2.$

25. $x^2+y^2+2(x+y)=12,$
$xy-(x+y)=2.$

26. $x^2+xy+y^2=21,$
$x-\sqrt{xy}+y=3.$

SUGGESTION. In Ex. 26 divide (1) by (2), obtaining $x+y+\sqrt{xy}=7$. Add this equation to (2) and find x in terms of y.

27. $x^2+tx+t^2=133,$
$t+x-\sqrt{tx}=7.$

28. $x^2+2xy+7y^2=24,$
$2x^2-xy-y^2=8.$

29. $(2x-3)(3y-2)=0,$
$4x^2+12xy-3y^2=0.$

SUGGESTION. In Ex. 29, from equation (1), $x=\frac{3}{2}$. The corresponding value of y is found by substituting this value of x in equation (2).

30. $x(x-y)=0,$
$x^2+2xy+y^2=9.$

31. $x-y-\sqrt{x-y}=2,$
$x^3-y^3=2044.$

SUGGESTION. In the first equation put $\sqrt{x-y}=z$, and solve for z, finding

$$x-y=1, \text{ or } 4.$$

Then from the second equation,

$$x^2+xy+y^2=2044, \text{ or } 511.$$

Put $y=x-1$ in $x^2+xy+y^2=2044$, and solve for x.

Similarly in the case of $x^2+xy+y^2=511$.

32. $3x^2-4y^2=8,$
$5(x-k)-4y=0.$

For what values of k are the solutions real? Imaginary? Equal?

33. $\sqrt{x}-\sqrt{y}=2,$
$(\sqrt{x}-\sqrt{y})\sqrt{xy}=30.$

34. Two men, A and B, dig a trench in 20 days. It would take A alone 9 days longer to dig it than it would B. How long would it take A and B each working alone?

35. A man spends \$539 for sheep. He keeps 14 of the flock that he buys, and sells the remainder at an advance of \$2 per head, gaining \$28 by the transaction. How many sheep did he buy and what was the cost of each?

SUPPLEMENTARY WORK

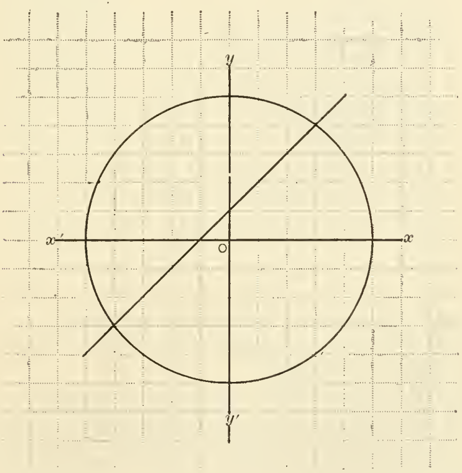
Graphs of Simultaneous Quadratic Equations

PREPARATORY.

1. In the same diagram construct graphs to represent the equations:

$$\begin{cases} x^2 + y^2 = 25, \\ x - y = -1. \end{cases}$$

Compare the result with this figure.



2. Solve the system of equations in Exercise 1.

Compare the values of x and y with the coördinates of the intersections of the graphs.

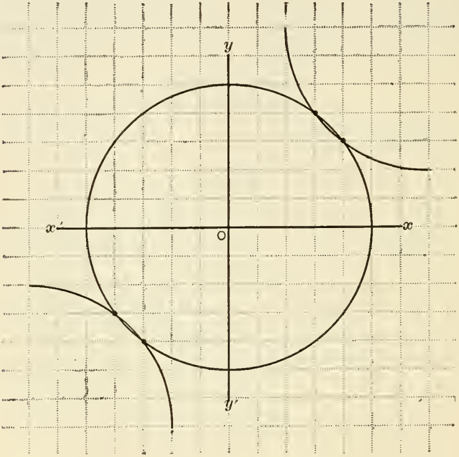
In how many points does the straight line intersect a circle?

How many solutions has the given system of equations?

3. Construct in one diagram the graphs of the equation:

$$\begin{cases} x^2 + y^2 = 25, \\ xy = 12. \end{cases}$$

Compare the result with this figure.



4. Solve the system of equations in Exercise 3.

Compare the values of x and y with the coördinates of the intersections of the graphs.

Graphical Solution of Two Simultaneous Equations. Every point of the graph of one of the equations has coördinates, x and y, that satisfy that equation; the points of intersection are points of both graphs and therefore have coördinates, x and y, that satisfy both equations. Hence, to solve two simultaneous equations graphically, draw their graphs and read the coördinates of their intersections. The coördinates of each point of intersection correspond to a solution. If the graphs do not intersect, the system of equations has no real roots.

WRITTEN EXERCISES

Solve graphically, and test by computing x and y:

1. $2x^2 + y = 1,$
   $x - y = 2.$

2. $x^2 - y^2 = 25,$
   $x + y = 1.$

3. $x^2 + y^2 = 10,$
   $xy = 3.$

4. $x^2 + y^2 = 13,$
   $xy = 6.$

5. $x^2 + y^2 = 13,$
   $x + 2y = 1.$

6. $x^2 - y^2 = 5,$
   $3x - y = 7.$

7. $x^2 + y^2 = 25,$
   $x^2 - y^2 = 7.$

8. $x^2 - 3xy + 2y^2 = 0,$
   $x^2 + 3y^2 = 16.$

9. $4x^2 + 9y^2 = 36,$
   $x^2 + y^2 = 25.$

10. $4x^2 + 9y^2 = 36,$
    $2x - 3y = 5.$

11. $4x^2 - 9y^2 = 36,$
    $x^2 + y^2 = 16.$

12. $4x^2 - 9y^2 = 36,$
    $xy = 18.$

CHAPTER IX

PROPORTION

174. Proportion. An equation between two ratios is called a **proportion.**

Thus, $\dfrac{a}{b}=\dfrac{c}{d}$ is a proportion.

The older form of writing this proportion is $a:b::c:d$; whence b and c are called the *means* and a and d the *extremes.* When so written, the proportion is read "a is to b as c is to d." At present, it is more customary to use the form $\dfrac{a}{b}=\dfrac{c}{d}$ and to read it "a over b equals c over d."

175. Fourth Proportional. In the proportion $\dfrac{a}{b}=\dfrac{c}{d}$, the fourth number, d, is called the **fourth proportional.**

176. Third Proportional. In the proportion $\dfrac{a}{b}=\dfrac{b}{c}$, the third number, c, although in the fourth place, is called the **third proportional** to a and b.

177. Mean Proportional. In the proportion $\dfrac{a}{b}=\dfrac{b}{c}$, b is called the **mean proportional** between a and c.

178. Relation to the Equation. A proportion is an equation and is to be treated in accordance with the properties of equations.

EXAMPLES

1. Find the fourth proportional to the numbers 6, 8, 30.

Let x be the fourth proportional, then, by Sec. 175, $\dfrac{6}{8}=\dfrac{30}{x}$. (*1*)

Multiplying by $8x$, $6x=8\cdot 30$. (*2*)

Dividing by 6, $x=40$. (*3*)

Therefore 40 is the fourth proportional to 6, 8, 30.

2. Find the third proportional to 5 and 17.

Let x be the third proportional, then, by Sec. 176, $\frac{5}{17}=\frac{17}{x}$. (*1*)

Multiplying both members by $17x$, $5x=17^2$. (*2*)

Solving (*2*), $x=\frac{289}{5}=57\frac{4}{5}$. (*3*)

Therefore $57\frac{4}{5}$ is the third proportional to 5 and 17.

WRITTEN EXERCISES

1. Write m, n, and p so that p shall be the third proportional to m and n.

2. Write m, n, and p so that n shall be the mean proportional between m and p.

Find x in each of the following proportions:

3. $-\frac{5}{x}=\frac{1}{4}$.

4. $\frac{x}{11}=\frac{7}{1331}$.

5. $x \div 16=\frac{1}{4} \div x$.

6. $-24 \div x=2x \div -12$.

7. $\frac{1.21}{x}=\frac{x}{.09}$.

8. $\frac{1}{6}=\frac{42}{x}$.

9. $\frac{75}{-x}=\frac{-x}{3}$.

10. Find the third proportional to 8 and $\sqrt{-5}$.

11. Find the third proportional to -6 and $\sqrt{3}$.

12. Find the mean proportionals between the following numbers: 9 and 16; -25 and -4; $\sqrt{-3}$ and $\sqrt{-7}$; 1 and -1.

13. If a sum of money earns \$48 interest in 5 yr., how much will it earn in 16 yr. at the same rate per cent?

14. A city whose population was 40,000 had 2500 school children; the total population increased to 48,000, and the number of children of school age increased proportionally. How many children of school age were there then?

15. What number must be added to each of the four numbers, 5, 29, 10, 44, to make the results proportional?

16. A lever need not be straight, although it must be rigid. Thus, the crank and the wheel and axle are varieties of the lever, and the law of the lever applies to them. Thus, in Fig. 1,

$$\frac{P}{W}=\frac{w}{p}.$$

Find W if $P=14$, $p=16$, and $w=4$.

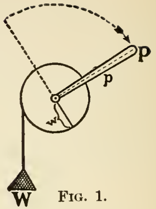


FIG. 1.

17. Find the unknown number:

	(1)	(2)	(3)
$P=$	$3a$	—	$a-b$
$W=$	$6b$	$5p$	$(a+b)^2$
$p=$	$2c$	$8p$	$a+b$
$w=$	—	$2p$	—

18. If an axle is 6 in. in diameter, what must be the diameter of the wheel in order that a boy exerting a force of 50 lb. may be able to raise 800 lb. weight?

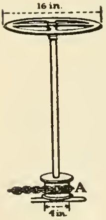


FIG. 2.

19. A brakeman pulls with a force of 150 lb. on a brake wheel 16 in. in diameter. The force is communicated to the brake by means of an axle, A, 4 in. in diameter. What is the pull on the brake chain?

20. Fig. 3 represents a weight W acting at P on an inclined plane, whose rate of slope is a vertical units to b horizontal units. It is known that the weight W acts in two ways: a force N pressing directly against the surface and tending to produce friction, and a force F parallel to the plane and tending to cause the weight to slide down the plane. It is known that these various quantities are related to each other thus:

$$\frac{F}{W}=\frac{a}{l}. \qquad \frac{N}{W}=\frac{b}{l}.$$

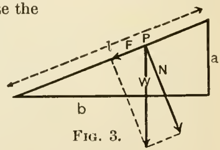


FIG. 3.

Find F and N, if $W=15$ lb., $a=20$ in., $b=21$ in., $l=29$ in.

21. Find l and N, if $F=16$ lb., $W=34$ lb., $a=4$ ft., $b=7\frac{1}{2}$ ft.

22. Find b and l, if $F=66$ lb., $N=112$ lb., $W=130$ lb., $a=33$ in.

23. Find W and a, if $F=200$ lb., $N=45$ lb., $b=9$, $l=41$.

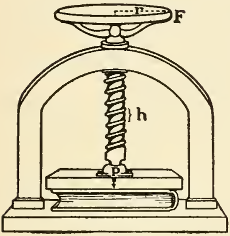


FIG. 1.

24. It is known that the pressure P exerted by a letter press on a book is connected with the force F and the other dimensions indicated in Fig. 1 by the following relation:

$$\frac{F}{P}=\frac{h}{2\pi r}.$$

Using $\frac{22}{7}$ as π, what pressure is exerted by a force of 20 lb. applied to the wheel of a letter press 16 in. in diameter and the threads of whose screw are $\frac{1}{2}$ in. apart?

25. It is known that if a circular arch of span s is raised h units in the middle, as indicated in Fig. 2, then h, s, r satisfy the proportion:

$$h:\frac{s}{2}=\frac{s}{2}:2r-h.$$

If a bridge in the form of a circular arch is to have a span of 200 ft. and be raised in the middle 6 ft. above the level of the end points, what is the radius of the circle?

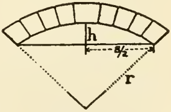
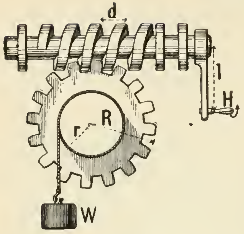


FIG. 2.

26. Find the unknowns:

	(1)	(2)	(3)	(4)
$h=$	10	7	——	——
$s=$	60	——	176	1470
$r=$	——	287	187	18015



27. When a force P is applied at H, the dimensions indicated in the figure and the weight W are known to be related as follows:

$$\frac{P}{W}=\frac{dr}{2\pi lR}.$$

If $R=18$ in., $r=10$ in., $l=12$ in., $d=\frac{1}{2}$ in., and $\frac{22}{7}$ be used for π, what weight can a man raise who turns the handle H with a force of 90 lb.?

28. In the formula of Exercise 27 find P, if $W=1000$ lb., $R=21$ in., $l=15$ in., $r=11$ in., $d=\frac{2}{3}$ in.

179. *In any proportion the product of the means equals the product of the extremes.*

For, $\frac{a}{b}=\frac{c}{d}$, and multiplying both members by bd, $ad=bc$.

180. Conversely,

If the product of two numbers equals the product of two other numbers, the four numbers can be arranged in a proportion, the two factors of one product being the means, and the two factors of the other product the extremes.

For, if $ad=bc$, divide both members by bd, and $\frac{a}{b}=\frac{c}{d}$.

181. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$.

For, if two numbers are equal, we know that their reciprocals are equal. See First Course, Sec. 193, p. 141.

The older form of statement that is still sometimes used is: If four numbers form a proportion, they are in proportion by *inversion*.

182. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

For, multiplying both members by $\frac{b}{c}$, $\quad \frac{b}{c}\cdot\frac{a}{b}=\frac{b}{c}\cdot\frac{c}{d}$, or,

canceling b and c, $\quad \frac{a}{c}=\frac{b}{d}$.

The older form of statement is: If four numbers form a proportion, they are in proportion by *alternation*.

183. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a \pm b}{b}=\frac{c \pm d}{d}$.

For, adding ± 1 to both members, $\frac{a}{b} \pm 1=\frac{c}{d} \pm 1$.

Therefore, performing the processes, $\frac{a \pm b}{b}=\frac{c \pm d}{d}$.

The older form of statement is: If four numbers form a proportion, they are in proportion by *composition* or by *division.*

184. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

For, from Sec. 183, $\frac{a+b}{b}=\frac{c+d}{d}$,

and $\frac{a-b}{b}=\frac{c-d}{d}$.

Dividing these equations, member for member,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

The older form of statement for this is: If four numbers form a proportion, they are in proportion by *composition* and *division.*

185. The **mean proportional** between two numbers is the square root of their product.

For, from $\frac{a}{b}=\frac{b}{c}$, Sec. 403, we find $b=\sqrt{ac}$.

186. The properties of proportion are useful in solving certain problems.

EXAMPLES

1. Solve: $$\frac{\sqrt{a+bx}+\sqrt{a-bx}}{\sqrt{a+bx}-\sqrt{a-bx}}=\frac{1}{2}. \qquad (1)$$

By Sec. 184, $$\frac{2\sqrt{a+bx}}{2\sqrt{a-bx}}=\frac{3}{-1}. \qquad (2)$$

Simplifying (2), $$-\sqrt{a+bx}=3\sqrt{a-bx}. \qquad (3)$$

Squaring (3), $$a + bx = 9a - 9bx. \quad (4)$$

Solving (4), $$10bx = 8a, \quad (5)$$

and $$x = \frac{4a}{5b}. \quad (6)$$

2. Given $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$. (1)

By Sec. 183, $$\frac{a+b}{b} = \frac{c+d}{d}. \quad (2)$$

$\therefore$ by Sec. 182, $$\frac{a+b}{c+d} = \frac{b}{d}. \quad (3)$$

Squaring both members of the given equation, $$\frac{a^2}{b^2} = \frac{c^2}{d^2}. \quad (4)$$

Applying Sec. 183 to (4), $$\frac{a^2+b^2}{b^2} = \frac{c^2+d^2}{d^2}. \quad (5)$$

Applying Sec. 182 to (5), $$\frac{a^2+b^2}{c^2+d^2} = \frac{b^2}{d^2}. \quad (6)$$

From (6), $$\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \frac{b}{d}. \quad (7)$$

Equating values of $\frac{b}{d}$ in (3) and (6), $$\frac{a+b}{c+d} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}. \quad (8)$$

WRITTEN EXERCISES

1. Given $\frac{a}{b} = \frac{b}{c}$, show that $\frac{a^2+b^2}{a+c} = \frac{a^2-b^2}{a-c}$.

2. Given $\frac{a}{b} = \frac{b}{c}$, show that $\frac{a^2-b^2}{a} = \frac{b^2-c^2}{c}$.

3. Given $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a}{d(a+b)^2} = \frac{c}{b(d+c)^2}$.

Solve the equations:

4. $\sqrt{x}+\sqrt{b} : \sqrt{x}-\sqrt{b}=a:b.$ 5. $\dfrac{x}{27}=\dfrac{y}{9}=\dfrac{2}{x-y}\cdot$

6. A passenger on an express train observes that a train going in the opposite direction passes him in 2 sec. If it had been going the same way, it would have passed in 30 seconds. What is the ratio of the rates of the two trains?

187. The following theorem concerning ratios is sometimes used:

In a series of equal ratios (a continued proportion), the sum of the numerators divided by the sum of the denominators is equal to any one of the ratios.

Proof: Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots$ be the equal ratios and r be their common value. (1)

Then $\frac{a}{b}=r, \frac{c}{d}=r, \frac{e}{f}=r$, and so on. (2)

$\therefore a=br,\ c=dr,\ e=fr$, and so on. (3)

$\therefore$ adding the equations in (3), $(a+c+e\cdots)=(b+d+f\cdots)r.$ (4)

$\therefore$ solving (4) for r, $\dfrac{a+c+e\cdots}{b+d+f\cdots}=r.$ (5)

$\therefore \dfrac{a+c+e\cdots}{b+d+f\cdots}=\dfrac{\text{the sum of the numerators}}{\text{the sum of the denominators}}=\dfrac{a}{b}$, or $\dfrac{c}{d}$, and so on, since r is any of these ratios. (6)

188. This theorem may be applied to certain equations.

EXAMPLES

Solve: $\dfrac{2x-2}{4-9x}=\dfrac{2}{9x}.$ (1)

Applying the theorem, $\dfrac{2x}{4}=\dfrac{2}{9x}.$ (2)

Simplifying (2), $\dfrac{x}{2}=\dfrac{2}{9x}.$ (3)

Solving (3), $9x^2=4,$ (4)

and $x=\pm\frac{2}{3}.$

Test by substitution.

WRITTEN EXERCISES

1. Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a+c-e}{b+d-f}=\frac{c}{d}$.

2. Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a^2+c^2+e^2}{b^2+d^2+f^2}=\frac{e^2}{f^2}$.

3. Also show that $\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}=\frac{a}{b}$.

4. Given $\frac{m}{n}=\frac{p}{q}=\frac{r}{s}$, show that $\frac{2m-3p+5r}{2n-3q+5s}=\frac{m}{n}$.

5. Given $\frac{x}{a}=\frac{b}{c}=\frac{e}{d}$, show that $x=\frac{a(b+e)}{c+d}$.

6. Find x in $\frac{x-7}{x+2}=\frac{5x+7}{3x-2}$.

SUMMARY

I. **Definitions.**

1. An equation between two ratios is called a *proportion.*

2. The numbers b and c are called the *means,* and a and d the *extremes.* Sec. 174.

3. In the proportion $\frac{a}{b}=\frac{c}{d}$, the fourth number d is called the *fourth proportional.* Sec. 175.

4. In the proportion $\frac{a}{b}=\frac{b}{c}$, the third number c is called the *third proportional.* Sec. 176.

5. In the proportion $\frac{a}{b}=\frac{b}{c}$, b is called the *mean proportional* between a and c, and is the square root of their product. Secs. 177 and 185.

II. **Properties.**

1. A proportion is an equation and therefore subject to the laws of equations. Sec. 178.

2. In any proportion the product of the means equals the product of the extremes. Sec. 179.

3. If the product of two numbers equals the product of two other numbers, the four numbers can be arranged in a proportion, the two factors of one product being the means, and those of the other product the extremes. Sec. 180.

4. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$ (inversion). Sec. 181.

5. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$ (alternation). Sec. 182.

6. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a \pm b}{b}=\frac{c \pm d}{d}$ (composition or division). Sec. 183.

7. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (composition and division). Sec. 184.

8. In a series of equal ratios the sum of the numerators divided by the sum of the denominators equals any one of the ratios. Sec. 187.

REVIEW

ORAL EXERCISES

1. What is the fourth proportional in $\frac{3}{3}=\frac{5}{d}$?

2. What is the third proportional in $\frac{3}{5}=\frac{5}{x}$?

3. What is the mean proportional between 4 and 9?

4. State the principle according to which it follows from

$$\frac{a}{b}=\frac{c}{d} \text{ that } \frac{a+b}{b}=\frac{c+d}{d}.$$

5. What is the value of x in $\frac{x}{a}=\frac{b}{x}$?

WRITTEN EXERCISES

Find the proportional to:

1. x, y, ab.   2. p^2, q, m^2q.   3. x^2, y^2, z^2.

Find the mean proportional between:

4. a^2, a^2b^2. **5.** -6, $4a^2b^2$. **6.** $27x^2y^3$, $3y$.

When $\frac{a}{b}=\frac{c}{d}$ show that:

7. $\frac{ac}{bd}=\frac{a^2}{b^2}$.

8. $\frac{a^2}{c^2}=\frac{a^2-b^2}{c^2-d^2}$.

9. $\frac{a+2b}{2b}=\frac{c+2d}{2d}$.

10. $\frac{2a-3b}{3b}=\frac{2c-3d}{3d}$.

11. $\frac{4a^2-5b^2}{5b^2}=\frac{4c^2-5d^2}{5d^2}$.

12. $\frac{a-b}{c-d}=\frac{\sqrt{a^2-b^2}}{\sqrt{c^2-d^2}}$.

13. The ratio of two numbers is 3 to 5 and the sum of their squares is 544. Find the numbers.

14. The lengths of the sides of a triangle are in the ratio of 3 to 4 to 5 and their sum is 120 ft. What is the length of each side?

SUPPLEMENTARY WORK

Adding the same positive number to both numerator and denominator of a ratio,

(*1*) *Increases the absolute value of the ratio if the numerator is less than the denominator; and*

(*2*) *Diminishes the absolute value of the ratio if the numerator is greater than the denominator.*

For, let $\frac{a}{b}$ be the given ratio and x the given number.

To determine whether $\frac{a}{b}$ or $\frac{a+x}{b+x}$ is the greater, subtract the second ratio from the first.

Then, $\frac{a}{b} - \frac{a+x}{b+x} = \frac{ab+ax-ab-bx}{b(b+x)} = \frac{(a-b)x}{b(b+x)}$.

(1) If $a > b$, the last ratio is a positive number, therefore the ratio $\frac{a}{b}$ would be diminished by adding x to each of its terms, which proves (*2*).

(2) If $a < b$, the last ratio is a negative number, therefore the ratio $\frac{a}{b}$ would be increased by adding x to each of its terms, which proves (*1*).

(3) If $a = b$, the last ratio is zero, therefore the ratio $\frac{a}{b}$ would be unchanged by adding x to each of its terms.

ORAL EXERCISES

1. How is the value of $\frac{2}{3}$ changed by adding 4 to each term?
2. How is the value of $\frac{3}{2}$ changed by adding 5 to each term?
3. How is the value of $\frac{4}{5}$ changed by adding a to each term?
4. How is the value of $\frac{5}{4}$ changed by adding b to each term?
5. What is the effect on $\frac{6}{6}$ of adding c to each term?

CHAPTER X

VARIATION

189. Direct Variation. When two variable quantities vary so as always to remain in the same ratio, each is said to **vary directly** as the other. Each increases or decreases at the same rate that the other increases or decreases.

Consequently, if x and y are two corresponding values of the variables and k the fixed ratio, then

$$\frac{x}{y} = k \text{ and } x = ky.$$

For example, at a fixed price (k) per article, the total cost (x) of a number of articles of the same sort varies directly with the number of articles (y). That is, $\frac{x}{y} = k$.

Likewise in the case of motion at a uniform rate (r), the distance traversed (d) varies as the time of motion (t). That is, $\frac{d}{t} = r$.

190. A symbol still occasionally used for "varies as" is $\propto$

Thus, "x varies as y" is written $x \propto y$,
and "d varies as t" is written $d \propto t$.

191. Relation of Variation to Proportion. When one variable varies directly as another, any pair of values of the variables forms a proportion with any other pair.

For, $\frac{x'}{y'} = r$ and $\frac{x''}{y''} = r$; $\therefore$ $\frac{x'}{y'} = \frac{x''}{y''}$, which is a proportion.

192. Expressions for Direct Variation. We have thus seen that the relation *x varies directly as y* may be expressed in any one of three ways:

(*a*) $x = ky$, by use of the equation.

(*b*) $x \propto y$, by use of the symbol of variation.

(*c*) $\frac{x'}{y'} = \frac{x''}{y''}$, by use of the proportion; x', y' and x'', y'' being any two pairs of corresponding values of the variables.

WRITTEN EXERCISES

1. Write the statement "v varies as w" in the form of an equation, also in the form of a proportion.

2. Write the statement $x = ky$ by use of the symbol $\propto$; also in the form of a proportion.

3. Write $\frac{t'}{v'} = \frac{t''}{v''}$ by use of the symbol $\propto$.

4. The weight (w) of a substance varies as the volume (v) when other conditions are unchanged. Express this law by use of the equation. By use of the symbol $\propto$. Also in the form of a proportion.

5. In the equation $w = kv$, if $w = 4$ and $v = 2$, what is the value of k? Using this value of k, what is the value of w when $v = 25$?

6. When 1728 cu. in. of a substance weigh 1000 ounces, what is the ratio of the weight (w) to the volume (v)? What volume of this substance will weigh 5250 ounces?

7. The cost (c) of a grade of silk varies as the number of yards (n). Find the ratio (r) of c to n when c is \$7.00 and $n = 4$.

8. If in Exercise 7, $\frac{c}{n} = r$, what does $\frac{c'}{n'}$ equal? Given that 40 yd. of silk cost \$60, find the cost of 95 yd. by means of the proportion $\frac{c'}{n'} = \frac{c}{n}$. Also by means of the equation $c = nr$.

193. Inverse Variation. A variable x is said to **vary inversely** as a variable y, if it varies directly as $\frac{1}{y}$.

Inverse variation means that when one variable is doubled the other is halved; when one is trebled the other becomes $\frac{1}{3}$ of its original value, and so on.

194. Expressions for Inverse Variation. The relation "x varies inversely as y" may be expressed:

(a) By the equation, $x = k\left(\frac{1}{y}\right)$, or $x = \frac{k}{y}$, $\therefore\ xy = k$.

(*b*) With the symbol of variation, $y \propto \frac{1}{x}$.

(*c*) As a proportion $\frac{x'}{x''} = \frac{y''}{y'}$.

WRITTEN EXERCISES

1. Write the statement "v varies inversely as w" in the form of an equation. Also in the form of a proportion.

2. Write $t = \frac{k}{p}$ by use of the symbol $\propto$, also in the form of a proportion.

3. In a bicycle pump the volume (v) of air confined varies inversely as the pressure (p) on the piston. Write the relation between v and p in three ways.

4. In Exercise 3, if $v = 18$ (cu. in.) and $p = 15$ (lb.), what is k in $v = \frac{k}{p}$? What is the pressure (p) when $v = 1$ (cu. in.)?

5. In an auditorium whose volume (v) is 25,000 cu. ft. there are 2000 persons (p). What is the number (n) of cubic feet of air space to the person? What will be the number when 1000 more persons come in?

6. The area of a triangle varies as the base times the altitude. If the area is 12 when the base is 8 and the altitude 3, what is the area of a triangle whose base is 40 and altitude 20?

7. The area of a circle varies as the square of its radius. The area of a circle of radius 2 is 12.5664; what is the area of a circle whose radius is 5.5?

8. The volume of a sphere varies as the cube of its radius. If the volume of a sphere whose radius is 3 is 113.0976, what is the volume of a sphere whose radius is 5?

9. If $x \propto y$ and $x = 6$ when $y = 2$, find x when $y = 8$.

SUGGESTION. $x = ky$. $\therefore 6 = k \cdot 2$ and $k = 3$. Substitute $y = 8$ in the equation $x = 3y$.

10. Determine k in $x \propto y$, if $x = 10$ when $y = 20$. Also if $x = 1$ when $y = 5$. If $x = 100$ when $y = 10$.

11. If $x \propto w$ and $y \propto w$, prove that $x + y \propto w$.

12. If $x \propto w$ and $w \propto y$, prove that $xy \propto w^2$.

13. If $x \propto y$ and $w \propto z$, prove that $\frac{x}{w} \propto \frac{y}{z}$.

14. If $x \propto y^2z$ and $x = 1$ when $y = 2$ and $z = 3$, find the constant k.

15. Given $y = z + w$, $z \propto x$ and $w \propto x$; and that $x = 1$ when $w = 6$, and that $x = 2$ when $z = 20$. Express y in terms of x.

SUGGESTION. $y = kx + k'x$; determine k and k'.

16. Given $z \propto x + y$ and $y \propto x^2$, and that $x = \frac{1}{2}$ when $y = \frac{1}{3}$ and $z = \frac{1}{4}$. Express z in terms of x.

GRAPHICAL WORK

195. The relation "x varies as y" has been expressed by the linear equation $x = ky$, and has been represented by *a straight line* (see FIRST COURSE, Secs. 212 and 213, pp. 168–170).

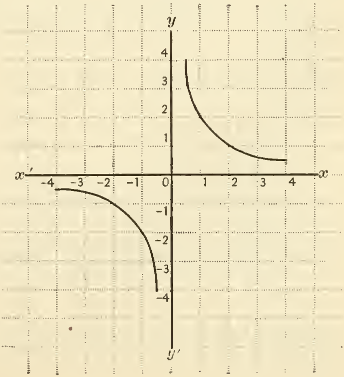
The relation "x varies inversely as y" has been expressed by the equation $x = \frac{k}{y}$. The graph of this equation is a *curve*.

Suppose, for illustration, that $k = 2$. Then $x = \frac{2}{y}$. For this equation:

The table is

x	y
5	$\frac{2}{5}$
4	$\frac{1}{2}$
3	$\frac{2}{3}$
2	1
1	2
$\frac{1}{2}$	4
-1	-2
-2	-1
-3	$-\frac{2}{3}$
-4	$-\frac{1}{2}$
-5	$-\frac{2}{5}$

The graph is



WRITTEN EXERCISES

Represent graphically:

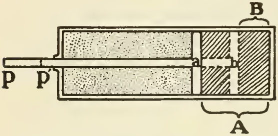
1. $x = \frac{1}{y}$, or $xy = 1$.

2. $x = \frac{-2}{y}$, or $xy = -2$.

3. $x = \frac{4}{y}$, or $xy = 4$.

4. $3x = \frac{-1}{y}$, or $3xy = -1$.

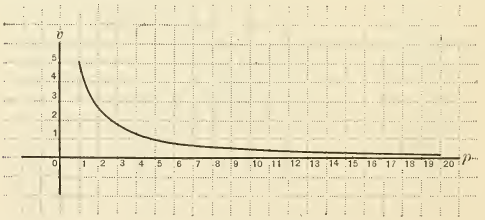
5. Under standard conditions the volume (v) of a confined gas varies inversely as the pressure (p). That is, $v = \frac{k}{p}$. This is known as Boyle's Law. Suppose when the piston is at a in the figure, the volume of the gas in part A is 1 cu. ft., and the pressure at P is 5 lb. When the pressure is 10 lb. the piston moves to b, and the volume of the gas B becomes $\frac{1}{2}$ cu. ft. What will be the volume of the gas when $P = 20$ lb.?



Taking $k = 5$ in the equation $v = \frac{\overset{5}{k}}{p}$, the table of values is

$p =$	1	2	3	4	5	6	7	8	9	10	15	20
$v =$	5	$2\frac{1}{2}$	$1\frac{2}{3}$	$1\frac{1}{4}$	1	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

The graph of the table is:



Read from the graph the pressure on the piston necessary to hold this volume of gas at $\frac{1}{2}$ cu. ft.; at $\frac{1}{4}$ cu. ft.; at 5 cu. ft.; at 2 cu. ft.

6. The attraction or "pull" of the earth on bodies in its neighborhood is the cause of their weight. The law of gravitation states that the weight of a given body varies inversely as the square of its distance from the center of the earth. This law may be expressed by the equation

$$w = \frac{k}{d^2}.$$

To construct the graph which shows the nature of the relation between w and d as they vary, k may be taken to be 1.

Fill the blanks in the table:

d	± 1	± 5	$\pm \frac{1}{3}$	$\pm .25$	$\pm .2$
w	1	( )	( )	( )	( )

7. Plot the graph for the table in Exercise 6.

SUMMARY

I. **Definitions.**

1. When two variable quantities vary so as always to remain in the same ratio, each *varies directly* as the other.
Sec. 189.

2. When one variable varies directly as another, any pair of values of the variables forms a proportion with any other pair.
Sec. 191.

3. A variable x *varies inversely* as a variable y, if it varies directly as $\frac{1}{y}$. Sec. 193.

II. **Processes.**

1. Processes involving direct variation are performed by reference to the equation $x = ky$. Sec. 192.

2. Processes involving inverse variation are performed by reference to the equation $x = \frac{k}{y}$. Sec. 194.

3. Direct and inverse variation may be represented graphically. Sec. 195.

SUPPLEMENTARY WORK

A large number of problems in science may be solved by the following plan:

EXAMPLES

1. The "law of gravitation" states that the weight of a given body varies inversely as the square of its distance from the center of the earth. What is the weight of a body 5 mi. above the surface of the earth, which weighs 10 lb. at the surface (4000 mi. from the center)?

METHOD. There are two variables in the problem, the weight (w) and the distance (d). There are also two parts or cases in the statement, one in which the value of one variable is unknown, and one in which the values of both variables are given.

Arrange the data as follows:

	w	d
1st case	x	4005
2d case	10	4000

To this table apply the law of variation expressed in the physical law. Since the law is: *w varies inversely as the square of d*, the values of d must be squared, and the ratio $x : 10$ equals the inverse ratio of 4005^2 to 4000^2; that is,

$$\frac{x}{10} = \frac{4000^2}{4005^2}.$$

$\therefore x = 9.96$, and the weight is 9.96 lb.

2. The squares of the times of revolution of the planets about the sun vary directly as the cubes of their distances from the sun. The earth is 93,000,000 mi. from the sun, and makes a revolution in approximately 365 da.; what is the distance of Venus from the sun, taking its time of revolution to be 226 da.?

SOLUTION

	t = time of revol.	d = distance from sun
1st case	365	93,000,000
2d case	226	x

According to the astronomical law the times must be squared and the distances cubed; then, since the law is that of direct variation:

$$\frac{365^2}{226^2} = \frac{93,000,000^3}{x^3}.$$

$\therefore x = 93,000,000 \sqrt[3]{\left(\frac{226}{365}\right)^2} = 68,900,000$, and the distance of Venus from the sun is approximately 69,000,000 miles.

WRITTEN EXERCISES

1. The intensity of light from a given source varies inversely as the square of the distance from the source. If the intensity (candle power) of the light from an electric lamp is 4 at a distance of 150 yd., what is its intensity at a distance of 25 yd.?

2. According to the first sentence of Exercise 1, how much farther from an electric light must a surface be moved to receive only $\frac{1}{4}$ as much light as formerly?

3. The time of oscillation of a pendulum varies directly as the square root of its length. What is the length of a pendulum which makes an oscillation in 5 sec., a 2-second pendulum being 156.8 in. long?

4. According to Exercise 3, what is the time of oscillation of a pendulum 784 in. long?

5. The distance through which a body falls from rest varies as the square of the time of falling. A body falls from rest 576 ft. in 6 sec.; how far does it fall in 10 sec.?

6. Volumes of similar solids vary as the cubes of their linear dimensions. The volume of a sphere of radius 1 in. is 4.1888 cu in.; what is the volume of a sphere whose radius is 5 in.?

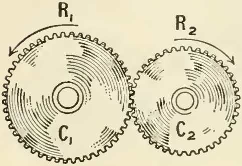
7. According to Exercise 6, what is the radius of a sphere whose volume is 33.5104 cu. ft.?

8. According to Exercise 6, if a flask holds $\frac{1}{2}$ pt., what is the capacity of a flask of the same shape 4 times as high?

9. In compressing a gas into a closed receptacle, as in pumping air into an automobile tire, the pressure varies inversely as the volume. If the pressure is 25 lb. when the volume is 125 cu. in., what is the pressure when the volume is 115 cu. in.?

10. According to Exercise 9, if the pressure is 50 lb. when the volume is 250 cu. in., what is the volume when the pressure is 10 lb.?

11. It is known that if one gear wheel turns another as in the figure, the number of revolutions of the two are to each other inversely as their number of teeth. That is, if the first has C_1 teeth and makes R_1 revolutions, and the second has C_2 teeth and makes in the same time R_2 revolutions,



then $$\frac{R_1}{R_2} = \frac{C_2}{C_1}.$$

Find R_2 if $C_1 = 25$, $C_2 = 15$, and $R_1 = 6$.

Find the numbers to fill the blanks:

	(1)	(2)	(3)	(4)
$C_1 =$	42	60	$5n$	—
$C_2 =$	—	50	$3n$	40
$R_1 =$	12	—	21	$8n$
$R_2 =$	9	15	—	$6n$

12. A diamond worth \$2000 was broken into two parts, together worth only \$1600. If the value of a diamond is proportional to the square of its weight, into what fractions was the original diamond broken? (Find result to nearest hundredth.)

13. If the volume of a sphere varies as the cube of its radius, find the radius of a sphere whose volume equals that of the sum of two spheres whose radii are respectively 6 ft. and 3.5 ft.

14. The number of vibrations (swings) made by two pendulums in the same time are to each other inversely as the square roots of their lengths. If a pendulum of length 39 in. makes 1 vibration per second (seconds pendulum), about how many vibrations will a pendulum 10 in. long make? How long must the pendulum be to make 10 vibrations per second?

15. Two towns join in building a bridge which both will use and agree to share its cost (\$ 5000) in direct proportion to their populations and in inverse proportion to their distances from the bridge. One town has a population of 5000 and is 2 mi. from the bridge; the other has a population of 9000 and is 6 mi. from the bridge. What must each pay?

16. If $a : b = p : q$, prove that

$$a^2 + b^2 : \frac{a^3}{a+b} = p^2 + q^2 : \frac{p^3}{p+q}.$$

SUGGESTION. The given proportion may be written:

$$\frac{a}{p} = \frac{b}{q}.$$

Let $\qquad \frac{a}{p} = r,$ then $a = pr,\ b = qr.$

The proportion to be proved may be written:

$$\frac{(a+b)(a^2+b^2)}{a^3} = \frac{(p+q)(p^2+q^2)}{p^3}.$$

Substituting the values of a and b above, the left member readily reduces to the right.

NOTE. A good method for proving such identities is to begin with the required relation and transform it into the given relation, or to transform both the given and the required relation until they reduce to the same thing.

17. Solve, using the principle of composition and division,

$$(a - \sqrt{2bx + x^2}) : (a - b) = (a + \sqrt{2bx + x^2}) : (a + b).$$

18. If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \cdots = \frac{a_n}{b_n}$, prove that

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{b_1 + b_2 + b_3 + \cdots + b_n} = \frac{a_1}{b_1}.$$

19. Given $x + \sqrt{x} : x - \sqrt{x} :: 3\sqrt{x} + 6 : 2\sqrt{x}$, to find x.

CHAPTER XI

SERIES

196. Series. If a set of numbers is specified according to some law or system, the set of numbers is called a **series.**

197. Terms. The numbers constituting the series are called its **terms**, and are named from the left, 1st term, 2d term, etc.

The following are examples of series:

1. $1, 2, 3, 4, 5, \cdots$.
2. $1, 3, 5, 7, 9, \cdots$.
3. $1, 5, 9, 13, 17, \cdots$.
4. $3, 6, 9, 12, 15, \cdots$.
5. $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \cdots$.
6. $2, 4, 8, 16, 32, \cdots$.
7. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots$.
8. $1, 3, 9, 27, 81, 243, \cdots$.
9. $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \cdots$.
10. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$.
11. $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \cdots$.
12. $100, 99, 98, 97, 96, 95, \cdots$.

ORAL EXERCISES

1–12. State the next five terms of each series above.

198. When the *law* of a series is known, any term may be found directly.

EXAMPLES

1. The law of the second series in Sec. 197 is that each term is two more than the preceding. To get the tenth term we start from 1 and add 2 for each of 9 terms. That is, the tenth term is $1 + 9 \cdot 2 = 19$.

Similarly, the 12th term is $1 + 11 \cdot 2 = 23$,
the 15th term is $1 + 14 \cdot 2 = 29$,
the 47th term is $1 + 46 \cdot 2 = 93$,
the nth term is $1 + (n - 1)2 = 2n - 1$.

2. The law of the eighth series is that each term after the first is 3 times the preceding term. To get the ninth term we start at 1 and multiply by 3 eight times, or by 3^8. That is, the ninth term is $1 \cdot 3^8 = 6561$.

Similarly, the 12th term is $1 \cdot 3^{11} = 177{,}147$,
the nth term is $1 \cdot 3^{n-1} = 3^{n-1}$.

WRITTEN EXERCISES

1–4. Select four of the series in Sec. 197 that can be treated like the first example and write the 10th, 12th, 15th, and 47th terms of each.

5–8. Select four of the series in Sec. 197 that can be treated like the second example and write the 8th, 10th, and nth terms of each.

9. Write similarly the 7th, 11th, 20th, 47th, and nth term for any 6 of the above series.

199. We shall give only two types of series, the arithmetical and the geometric, the laws of which are comparatively simple.

ARITHMETICAL SERIES

200. Arithmetical Series. A series in which each term after the first is formed by adding a fixed number to the preceding term is called an **arithmetical series** or **arithmetical progression.**

201. Common Difference. The fixed number is called the **common difference,** and may, of course, be negative.

For example:

1. 7, 15, 23, 31, 39, ··· is an arithmetical series having the common difference 8.

2. 16, $14\frac{1}{2}$, 13, $11\frac{1}{2}$, 10, ··· is an arithmetical series having the common difference $-\frac{3}{2}$.

ORAL EXERCISES

1. Name all the arithmetical series in the list, Sec. 197.
2. State the common difference in each of these series.
3. State the 10th term in each of these series.

WRITTEN EXERCISES

1. Write the nth term in each arithmetical series in the list, Sec. 197.
2. Beginning with 2 find the 100th even number.
3. Beginning with 1 find the 100th odd number.

4. Beginning with 3 find the 200th multiple of 3.

5. A city with a population of 15,000 increased 600 persons per year for 10 yr. What was the population at the end of 10 yr.?

202. A general form for an arithmetical series is:

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ a+(n-1)d,\ \cdots,$$

where a denotes the first term,
d denotes the common difference, and
n denotes the number of the term.

203. Last Term. If the last term considered is numbered n and denoted by l, we have for the last of n terms the formula: $l=a+(n-1)d$.

204. The Sum of an Arithmetical Series. The sum of n terms of an arithmetical series can be found readily.

EXAMPLE

Find the sum of the first 6 even numbers.

1. Let $s=2+4+6+8+10+12.$
2. We may also write $s=12+10+8+6+4+2.$
3. Adding (1) and (2),

$$2s=(2+12)+(4+10)+(6+8)+(8+6)+(10+4)+(12+2)$$

$=6(2+12)$; for each parenthesis is the same as $2+12$.

4. $$\therefore s=\frac{6(2+12)}{2}=42.$$

WRITTEN EXERCISES

Find similarly the sum of:

1. The first 6 odd numbers.
2. The first 6 multiples of 3.
3. The first 5 multiples of 7.
4. The first 4 multiples of 8.

205. General Formula for the Sum. The general form of the series may be treated in the same way.

If l denotes the last of n terms, the term before it is denoted

by $l-d$, the next preceding by $l-2d$, and so on. Hence, the sum of n terms may be written:

$$s=a+(a+d)+(a+2d)+\cdots+(l-2d)+(l-d)+l.$$

And also, $s=l+(l-d)+(l-2d)+\cdots+(a+2d)+(a+d)+a.$

Whence, adding, $2s=(a+l)+(a+l)+\cdots+(a+l)=n(a+l).$

Therefore, $$s=\frac{n(a+l)}{2}.$$

Or, in words,

The sum of any number of terms of an arithmetical series is one half the sum of the first and the last terms times the number of terms.

By using the value of l in Sec. 203, $s=\dfrac{n[2a+(n-1)d]}{2}.$

This permits the calculation of s without working out separately the value of l.

WRITTEN EXERCISES

For each series in the following list find: First, the sum of 10 terms. Second, the sum of n terms.

1. 1, 2, 3, 4, 5, ⋯.
2. 1, 3, 5, 7, 9, ⋯.
3. 1, 5, 9, 13, 17, ⋯.
4. 3, 6, 9, 12, 15, ⋯.
5. 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, ⋯.
6. 100, 99, 98, 97, 96, 95, ⋯.

7. A man invests $100 of his earnings at the beginning of each year for 10 yr. at 6%, simple interest. How much has he at the end of 10 yr.?

Solution.

1. The last investment bears interest 1 yr. and amounts to $106; the next to the last bears interest 2 yr. and amounts to $112, etc.; the first bears interest 10 yr. and amounts to $160.

2. Hence, $a=\$106$, $d=\$6$, and $n=10$.

3. Therefore, $l=106+9\cdot 6=160$.

4. Therefore, $s=\frac{10}{2}(106+160)=1330$.

5. The man has $1330.

8. If $50 is invested at the beginning of each year for 20 yr. at 5% simple interest, what is the amount at the end of 20 yr.?

9. If a body falls approximately 16 ft. the first second and 32 ft. farther in each succeeding second, how far does it fall in 5 sec.?

206. Collected Results. The three chief formulas of arithmetical series are:

1. $l = a + (n-1)d.$
2. $s = \dfrac{n(a+l)}{2}.$
3. $s = \dfrac{n[2a + (n-1)d]}{2}.$

GEOMETRIC SERIES

207. Geometric Series. A series in which each term after the first is formed by multiplying the preceding term by a fixed number is called a **geometric series**, or a **geometric progression**.

208. Common Ratio. The fixed multiplier is called the **common ratio**, and may be negative.

For example:

1. 2 is the common ratio in the geometric series 2, 4, 8, 16, $\cdots$.
2. $-\frac{1}{3}$ is the common ratio in the geometric series, 27, -9, 3, -1, $\frac{1}{3}$, $\cdots$.

ORAL EXERCISES

1. Name all the geometric series in the list, Sec. 197.
2. State the common ratio in each of these series.
3. State the 6th term of each of these series.

WRITTEN EXERCISES

1. The series 1, 3, 9, 27, 81, $\cdots$, whose ratio is 3, is the same as 1, 3^1, 3^2, 3^3, 3^4, $\cdots$. Write by use of exponents the 6th term of this series; the 8th term; the 10th term; the 15th; the 25th; the 100th.

2. The series 3, $-\frac{3}{2}$, $\frac{3}{4}$, $-\frac{3}{8}$, $\cdots$, whose ratio is $-\frac{1}{2}$, is the same as 3, $3(-\frac{1}{2})$, $3(-\frac{1}{2})^2$, $3(-\frac{1}{2})^3$, $\cdots$. Write by use of exponents the 5th term of this series; the 8th term; the 10th; the 25th; the 50th.

209. A general form for the geometric series is:

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1}\cdots,$$

where a denotes the first term,
r denotes the common ratio, and
n denotes the number of the terms.

210. Last Term. If the last term is numbered n, and denoted by l, then we have for the last of n terms the formula,

$$l = ar^{n-1}.$$

211. The Sum of a Geometric Series. The sum of n terms of a geometric series can readily be found.

EXAMPLE

Find the sum of 5 terms of the series 2, 6, 18, 54, 162.

SOLUTION.

Let	$s = 2 + 6 + 18 + 54 + 162.$	(1)
Multiplying by 3, the common ratio,	$3s = 6 + 18 + 54 + 162 + 486.$	(2)
Subtracting (1) from (2),	$3s - s = 486 - 2,$	(3)
or,	$2s = 484.$	(4)
Dividing by 2,	$s = 242.$	(5)

WRITTEN EXERCISES

Find similarly the sum of 5 terms of each of these series:

1. $6, 30, 150, \cdots.$
2. $7, -14, 28, \cdots.$
3. $\frac{1}{2}, \frac{1}{10}, \frac{1}{50}, \cdots.$
4. $\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, \cdots.$

212. General Formula for the Sum. The general form of the series may be treated in the same way. If l denote the last of n terms, the term before it is denoted by $\frac{l}{r}$, the next preceding by $\frac{l}{r^2}$, and so on. Hence, the sum of n terms may be written:

$$1.\quad s = a + ar + ar^2 + \cdots \frac{l}{r^2} + \frac{l}{r} + l.$$

2. Then $rs = ar + ar^2 + \cdots \frac{l}{r^2} + \frac{l}{r} + l + lr.$

3. Subtracting, $s - rs = a - lr.$

4. Or, $(1 - r)s = a - lr.$

$$\therefore s = \frac{a - lr}{1 - r} = \frac{lr - a}{r - 1}.$$

In words,

The sum of any number of terms of a geometric series is the ratio times the last term diminished by the first term and divided by the ratio less 1.

By using the value of l (Sec. 210),

$$s = \frac{ar^{n-1} \cdot r - a}{r - 1} = \frac{ar^n - a}{r - 1}.$$

Thus, s may be found without first computing l.

WRITTEN EXERCISES

1. Find the sum of 6 terms of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.
2. Find the sum of 10 terms of the series $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \cdots$.
3. Find the sum of 8 terms of the series 1, .25, .0625, $\cdots$.
4. Find the sum of 12 terms of the series $27, -9, 3, -1, \cdots$.
5. An air pump exhausted the air from a cylinder containing 1 cu. ft. at the rate of $\frac{1}{10}$ of the remaining contents per stroke. What part of a cubic foot of air remained in the cylinder after 25 strokes?
6. The population of a town increased from 10,000 to 14,641 in 5 yr. If the population by years was in geometric series, what was the rate of increase per year?

213. Collected Results. The three chief formulas of geometric series are:

1. $l = ar^{n-1}.$
2. $s = \dfrac{lr - a}{r - 1}.$
3. $s = \dfrac{ar^n - a}{r - 1}.$

WRITTEN EXERCISES

1. \$100 is placed on interest at 5 %, compounded annually.

(1) What is the amount at the end of the first year?

(2) What is the principal for the second year?

(3) What is the amount at the end of the second year?

1st	\$100 (1.05)
2d	\$100 $(1.05)^2$
3rd	\$100 $(1.05)^3$
4th	\$100 $(1.05)^4$

Notice that the amounts appear in the right-hand column of the table.

(4) Indicate similarly the amount of \$100 at the end of 5 yr.; 10 yr.; n yr. Which formula of geometric series expresses the amount for n yr.?

2. Indicate the amount of \$100 at 6 %, compounded annually, at the end of 1 yr.; 2 yr.; 5 yr.; 10 yr.; n yr.

3. Many savings banks pay interest at the rate of 3 %, compounded semiannually.

Indicate the amount of \$100 under the above conditions at the end of 6 mo.; 1 yr.; 18 mo.; 2 yr.; 10 yr.; n yr.

Note. The numerical value of these expressions can be computed readily by logarithms.

Solve by the use of logarithms:

4. What is the amount of \$1 at 4 % compound interest for 8 yr.?

5. A man deposits \$100 in a bank paying 4 % interest, compounded annually, on the first day of each year for 5 yr. How much will he have on deposit at the end of 5 yr.?

6. Determine similarly the amounts of:

	Deposit at Beginning of Each Year	Rate of Interest Compounded Annually	Number of Years
(1)	\$25	5	8
(2)	10	6	15
(3)	43.20	4	20
(4)	39.87	3	20

MEANS

214. Means. Terms standing between two given terms of a series are called **means**.

215. Arithmetical Mean. If three numbers are in arithmetical series, the middle one is called the **arithmetical mean** between the other two.

The arithmetical mean between a and b is found thus:

1. Let A be the mean and d the common difference.
   Then the terms may be written $A - d$ and $A + d$.
2. Whence, $A - d = a$ and $A + d = b$.
3. Adding, $2A = a + b$ and $A = \frac{a+b}{2}$.

216. *The arithmetical mean between two numbers is one half their sum.*

217. Geometric Mean. If three numbers are in geometric series, the middle one is called the **geometric mean** between the other two.

The geometric mean between a and b is found thus:

1. Let g be the geometric mean.
2. Then $\frac{g}{a} = \frac{b}{g}$.
3. $\therefore g^2 = ab$ and $g = \sqrt{ab}$.

218. *The geometric mean between two numbers is the square root of their product. There are really two geometric means, one negative and one positive.*

The geometric mean between two numbers is the same as their mean proportional.

ORAL EXERCISES

State the arithmetical mean between:

1. 8, 12. **2.** 6, 3. **3.** 4, -10. **4.** $5a$, $13a$.

State the geometric mean, including signs, between:

5. 8, 6. **6.** 3, 12. **7.** a, a^5. **8.** $2x^3$, $32x^7$.

219. Any number of means may be found by use of formulas already given.

EXAMPLES

1. Insert 5 arithmetical means between 4 and 12.

1. In this case $a = 4$, $l = 12$, and $n = 7$.
2. $\therefore l = a + (n - 1)d$ becomes $12 = 4 + (7 - 1)d$.
3. Solving for d, $d = \frac{12 - 4}{6} = 1\frac{1}{3}$.
4. Adding $1\frac{1}{3}$ to 4, and $1\frac{1}{3}$ to that result, and so on, the means are found to be $5\frac{1}{3}$, $6\frac{2}{3}$, 8, $9\frac{1}{3}$, and $10\frac{2}{3}$.

2. Insert 4 geometric means between -27 and $\frac{1}{9}$.

1. In this case $a = -27$, $l = \frac{1}{9}$, and $n = 6$.
2. $\therefore l = ar^{n-1}$ becomes $\frac{1}{9} = -27r^5$.
3. Solving for r, $r^5 = -\frac{1}{9 \cdot 27} = -\frac{1}{3^5}$.

Therefore, $r = -\frac{1}{3}$.

4. Multiplying -27 by $-\frac{1}{3}$, and multiplying this result by $-\frac{1}{3}$, and so on, the means are found to be 9, -3, 1, and $-\frac{1}{3}$.

WRITTEN EXERCISES

1. Insert 3 arithmetical means between 6 and 26.
2. Insert 10 arithmetical means between -7 and 144.
3. Insert 3 geometric means between 2 and 32.
4. Insert 4 geometric means between $-\frac{1}{10}$ and $3\frac{1}{5}$.

OTHER FORMULAS

220. Arithmetical Series. By means of the formulas of Sec. 206, any two of the five numbers, a, n, l, d, s, can be found when the other three are given.

EXAMPLES

1. Given $n = 6$, $s = 18$, $l = 8$, find a, d.

1. For these values, formulas (1) and (2) become:

$$8 = a + (6 - 1)d,$$

$$18 = \frac{6(a + 8)}{2}.$$

2. We have thus two equations to determine the two numbers, a, d.
   From the second equation, $a = -2$.
3. Using this value in the first equation, $d = 2$.

2. Given $a = 4$, $l = 12$, $s = 56$, find n and d.

   1. For these values formulas (1) and (2) become:
   $$12 = 4 + (n - 1)d.$$
   $$56 = \frac{n(4 + 12)}{2}.$$
   2. $\therefore$ from the second equation, $n = 7$.
   3. Substituting in the first, $12 = 4 + 6\,d$,
   4. therefore $d = \frac{4}{3}$.

3. Given $n = 12$, $s = 30$, $l = 10$, find a, d.

   1. Formulas (1) and (3) become:
   $$10 = a + 11\,d.$$
   $$30 = \frac{12(2\,a + 11\,d)}{2} = 12\,a + 66\,d.$$
   2. Solving these equations for a and d:
   $$a = -5, \text{ and } d = \tfrac{15}{11}.$$

The same results would be found by using formulas (1) and (2), since (3) is only another form of (2).

221. The same problems can also be solved generally; that is, without specifying numerical values.

EXAMPLE

Regarding n, s, d as known, find a, l.

1. From (1), Sec. 206, $\quad l - a = (n - 1)d.$
2. From (2), Sec. 206, $\quad a + l = \frac{2\,s}{n}.$
3. Adding (1) and (2), $\quad 2\,l = (n - 1)\,d + \frac{2\,s}{n}.$

   Or, $\quad l = \frac{(n - 1)d}{2} + \frac{s}{n}.$
4. Substituting in (2) above, $\quad a = \frac{2\,s}{n} - \left[\frac{(n - 1)d}{2} + \frac{s}{n}\right]$
   $$= \frac{s}{n} - \frac{(n - 1)d}{2}.$$

WRITTEN EXERCISES

By use of the formulas in Sec. 206, find the following:

	Find	In Terms of	Result
1.	l	$a\ d\ n$	$l = a + (n-1)d$
2.	l	$a\ d\ s$	$l = \frac{1}{2}[-d \pm \sqrt{8\,ds + (2\,a - d)^2}]$
3.	l	$a\ n\ s$	$l = \frac{2\,s}{n} - a$
4.	l	$d\ n\ s$	$l = \frac{s}{n} + \frac{(n-1)d}{2}$
5.	s	$a\ d\ n$	$s = \frac{1}{2}\,n[2\,a + (n-1)d]$
6.	s	$a\ d\ l$	$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2\,d}$
7.	s	$a\ n\ l$	$s = \frac{n}{2}(a + l)$
8.	s	$d\ n\ l$	$s = \frac{1}{2}\,n\,[2\,l - (n-1)d]$
9.	a	$d\ n\ l$	$a = l - (n-1)d$
10.	a	$d\ n\ s$	$a = \frac{s}{n} - \frac{(n-1)d}{2}$
11.	a	$d\ l\ s$	$a = \frac{1}{2}\,[d \pm \sqrt{(2\,l + d)^2 - 8\,ds}]$
12.	a	$n\ l\ s$	$a = \frac{2\,s}{n} - l$
13.	d	$a\ n\ l$	$d = \frac{l-a}{n-1}$
14.	d	$a\ n\ s$	$d = \frac{2\,(s - an)}{n(n-1)}$
15.	d	$a\ l\ s$	$d = \frac{l^2 - a^2}{2\,s - l - a}$
16.	d	$n\ l\ s$	$d = \frac{2(nl - s)}{n(n-1)}$
17.	n	$a\ d\ l$	$n = \frac{l-a}{d} + 1$
18.	n	$a\ d\ s$	$n = \frac{d - 2\,a \pm \sqrt{(2\,a - d)^2 + 8\,ds}}{2\,d}$
19.	n	$a\ l\ s$	$n = \frac{2\,s}{l + a}$
20.	n	$d\ l\ s$	$n = \frac{2\,l + d \pm \sqrt{(2\,l + d)^2 - 8\,ds}}{2\,d}$

Note. a, l, d, s may have any values, but n must be a positive integer. Hence, when n is one of the unknowns, not all the solutions that satisfy the equations will correspond to a possible arithmetical series.

222. Geometric Series. By means of the formulas of Sec. 213, any two of the five numbers, a, n, l, r, s, can be found when the other three are given.

EXAMPLES

1. Given $s = 1024$, $r = 2$, $a = 2$, find l.

   1. For these values formula (2) becomes:

   $$1024 = \frac{2l - 2}{2 - 1} = 2(l - 1).$$

   2. $\therefore\ l = 513$.

2. Given $r = 3$, $n = 5$, $s = 363$, find a.

   1. For these values formulas (1) and (2) become:

   $$l = a \cdot 3^4.$$
   $$363 = \frac{3l - a}{2}.$$

   2. Eliminating l, $\quad 363 = \dfrac{a(3^5 - 1)}{2}.$

   3. Therefore, $\quad 363 = \dfrac{a \cdot 242}{2}$, and $a = 3$.

3. Given $s = 363$, $a = 3$, $r = 3$, find n.

   1. For these values formula (3) becomes:

   $$363 = \frac{3 \cdot 3^n - 3}{2}.$$

   2. Therefore, $3^n - 1 = 242$, and $3^n = 243$.
   3. By factoring 243, n is seen to be 5.

NOTE. In finding n it may not be possible to factor as in the case of 243 above. In this case logarithms may be applied.

223. The same problems can be solved generally, that is, without specifying numerical values.

EXAMPLE

Express l in terms of a, n, and s.

1. From formula (2) $r = \dfrac{s - a}{s - l}$, or, $r^{n-1} = \dfrac{(s - a)^{n-1}}{(s - l)^{n-1}}$.

2. $\therefore$ substituting in (1) $l = \dfrac{a(s - a)^{n-1}}{(s - l)^{n-1}}$.

3. $\therefore\ l(s - l)^{n-1} - a(s - a)^{n-1} = 0$.

This equation is of a degree higher than 2 in l when $n > 3$. But for n equal to or less than 3 it can be solved by methods already explained.

WRITTEN EXERCISES

By use of the formulas in Sec. 213, find the following:

NOTE. In Exercises 3, 12, and 16, only the equation connecting the unknown numbers with the given ones can be found:

	FIND	IN TERMS OF	RESULT
1.	l	$a\ r\ n$	$l = ar^{n-1}$
2.	l	$a\ r\ s$	$l = \dfrac{a + (r-1)s}{r}$
3.	l	$a\ n\ s$	$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$
4.	l	$r\ n\ s$	$l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}$
5.	s	$a\ r\ n$	$s = \dfrac{a(r^n - 1)}{r-1}$
6.	s	$a\ r\ l$	$s = \dfrac{rl - a}{r-1}$
7.	s	$a\ n\ l$	$s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$
8.	s	$r\ n\ l$	$s = \dfrac{lr^n - l}{r^n - r^{n-1}}$
9.	a	$r\ n\ l$	$a = \dfrac{l}{r^{n-1}}$
10.	a	$r\ n\ s$	$a = \dfrac{(r-1)s}{r^n - 1}$
11.	a	$r\ l\ s$	$a = rl - (r-1)s$
12.	a	$n\ l\ s$	$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$
13.	r	$a\ n\ l$	$r = \sqrt[n-1]{\dfrac{l}{a}}$
14.	r	$a\ n\ s$	$r^n - \dfrac{s}{a}r + \dfrac{s-a}{a} = 0$
15.	r	$a\ l\ s$	$r = \dfrac{s-a}{s-l}$
16.	r	$n\ l\ s$	$r^n - \dfrac{s}{s-l}r^{n-1} + \dfrac{l}{s-l} = 0$

SUMMARY

I. **Definitions.**

1. A set of numbers specified according to some law is called a *series*. The numbers constituting a series are called its *terms*. Secs. 196, 197.

2. If each term of a series after the first is found by adding a fixed number (*common difference*) to the preceding term, the series is called an *arithmetical series*, or *arithmetical progression*; if each term after the first is found by multiplying the preceding term by a fixed number (*common ratio*), the series is called a *geometric series*, or *geometric progression*. Secs. 200, 201, 207, 208.

3. If three numbers are in arithmetical (or geometric) series, the middle one is called the arithmetical (or geometric) *mean* between the other two. Secs. 440–444.

II. **Notations.** a = first term.
d = common difference.
r = common ratio.
n = number of a term, or number of terms considered.
l = last (of n terms).
s = sum (of n terms).

III. **Important Formulas.**

Arithmetical Series	Geometric Series
$l = a + (n-1)d$	$l = ar^{n-1}$
$s = \dfrac{n(a+l)}{2}$	$s = \dfrac{lr - a}{r-1}$
$s = \dfrac{n[2a + (n-1)d]}{2}$	$s = \dfrac{ar^n - a}{r-1}$

REVIEW

WRITTEN EXERCISES

1. Find the 47th multiple of 7.
2. Find the sum of the first 12 multiples of 4.

Find the 20th term, and the sum of 12 terms of each series:

3. $6, 9, \frac{27}{2}, \frac{81}{4}, \cdots$.

4. $8, 11, 14, 17, \cdots$.

5. $2^9, 2^6, 2^3, \cdots$.

6. $a + b, a - b, a - 3b, a - 5b$.

Find the eighth term, and the sum of 8 terms:

7. $1, 4, 16, \cdots$.

8. $3, 6, 12, \cdots$.

9. $2, -4, 8, -16, \cdots$.

10. $1, -2, 2^2, -2^3, \cdots$.

11. $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \cdots$.

12. $100, -40, 16, \cdots$.

Find the twelfth term, and the sum of 12 terms:

13. $2, 4, 6, \cdots$.

14. $-5, -3, -1, \cdots$.

15. $1, \frac{6}{7}, \frac{5}{7}, \cdots$.

16. $\frac{2}{3}, \frac{7}{15}, \frac{4}{15}, \cdots$.

17. $4, -3, -10, \cdots$.

18. $\frac{1}{2}, -\frac{2}{3}, -\frac{11}{6}, \cdots$.

19. Find three numbers whose common difference is 1 and such that the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

20. The first term of an arithmetic series is $n^2 - n - 1$, the common difference is 2. Find the sum of n terms.

21. In Italy the hours of the day are numbered from 1 to 24. How many strokes would a clock make per day in striking these hours?

22. How many strokes does a common clock striking the hours make in a day?

23. A man leases a business block for 20 years under the condition that, owing to estimated increase in the value of the property, the rental is to be increased \$50 each year. He pays altogether \$19,500. What was the rental of the first year? The last?

24. A railroad car starting from rest began to run down an inclined plane. It is known that in such motion the distances passed over in successive seconds are in arithmetical progression. It was observed that at the end of 10 sec. the car had passed over 570 ft. and at the end of 20 sec. 2340 ft.

from the starting point. How far did it run the first second? How far from the starting point was it at the end of 15 sec.?

25. It is known that if a body falls freely, the spaces passed over in successive seconds are in arithmetical progression, and that it falls approximately 16 ft. in the first second and 48 ft. in the next second. To determine the height of a tower, a ball was dropped from the top and observed to strike the ground in 4 sec. Find the height of the tower.

26. An employee receives a certain annual salary, and in each succeeding year he receives \$72 more than the year before. At the end of the tenth year he had received altogether \$10,440. What was his salary the first year? The last?

27. The 14th term of an arithmetical series is 72, the fifth term is 27. Find the common difference and the first term.

28. A man is credited \$100 annually on the books of a building society as follows: At the beginning of the first year he pays in \$100 cash. At the beginning of the second year he is credited with \$6 interest on the amount already to his credit; and he is required to pay \$94 in cash, making his total credit \$200. At the beginning of the third year he is credited with \$12 interest, and pays \$88 in cash, and so on. How much is his payment at the beginning of the tenth year? What is his credit then? How much cash has he paid altogether?

29. At each stroke an air pump exhausts $\frac{2}{3}$ of the air in the receiver. What part of the original air remains in the receiver after the 8th stroke?

30. At the close of each business year, a certain manufacturer deducts 10 % from the amount at which his machinery was valued at the beginning of the year. If his machinery cost \$10,000, at what did he value it at the end of the fourth year?

31. By use of logarithms, find its valuation at the end of the 20th year.

SUPPLEMENTARY WORK

Special Series

Finite Series. So far we have treated only series with a fixed number of terms. A series which comes to an end is called a **finite series.**

Infinite Series. A series whose law is such that every term has a term following it is called an **infinite series.**

For example:

2, 5, 8, 11, ⋯, 239 as here written ends with 239. But the law of the series would permit additional terms to be specified. In the above example, the next following terms would be 242, 245, etc. It is obvious that however many terms may have been specified, still more can be made by adding 3. The series is thus unending.

Similarly, all of the series so far considered might have been continued by applying their corresponding laws.

The term "infinite" comes from the Latin *infinitus*, and is here used with the meaning, *unending.*

If the coefficients of the binomial expansion be regarded as a series,

$$1,\ n,\ \frac{(n-1)}{2\,!},\ \frac{n(n-1)(n-2)}{3\,!},\ \frac{n(n-1)(n-2)(n-3)}{4\,!},\ \cdots$$

they furnish, when n is a positive integer, instances of series that come to an end according to the law of the series. If $n = 3$, the series has 4 terms, and if $n = 10$, the series has 11 terms; for the positive integer n, it has $n + 1$ terms. This is true because the factors n, $n - 1$, $n - 2$, and so on, will finally in the $(n + 2)$nd term contain $n - n$ or zero. Therefore the series has $n + 1$ terms.

But if n is a negative integer or a fraction, none of the factors, n, $n - 1$, $n - 2$, and so on, become zero, and the series can always be extended farther. That is, if n is a negative integer or any fraction, the series is unending or infinite.

Infinite Geometric Series. The subject of infinite series is of great importance, but is far too difficult to be taken up here. We mention simply a few properties of infinite geometric series whose ratio is numerically less than 1.

The following are examples of such series:

1. $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \cdots$.
2. $3, \frac{3}{5}, \frac{3}{25}, \frac{3}{125}, \cdots$.
3. $.5, .05, .005, .0005, \cdots$.
4. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \cdots$.

State the ratio and the next three terms of each series.

I. *The terms become numerically smaller and smaller.* Each term is numerically smaller than the one preceding it, for it is a proper fraction of it.

II. *The terms become numerically small at will.* That is, however small a number may be selected, there are terms in the series smaller than it, and when r is numerically less than 1, the term ar^{n-1} may be made numerically small at will, by taking n sufficiently large.

This seems obvious from the consideration of the series given above as examples. It is not difficult to prolong these series until their terms are less than $\frac{1}{100}$ say, or $\frac{1}{1000}$, and from this it seems plausible to think that the terms would become less than one millionth, or one billionth, or any other number, if a sufficient number of terms are taken. As a matter of fact this is true, but the proof is too difficult to be given here.

III. We have proved that, if s_n denote the sum of the first n terms of a geometric series,

$$s_n = \frac{a - ar^n}{1 - r}.$$

This may be written: $s_n = \dfrac{a}{1-r} - ar^{n-1}\left(\dfrac{r}{1-r}\right).$

By taking n sufficiently large, the product of ar^{n-1} and the fixed number $\dfrac{r}{1-r}$ can be made as small as desired. As more and more terms of the series are added, the sum differs less

and less from $\frac{a}{1-r}$; and if sufficiently many terms are taken, the sum comes as close as we please to $\frac{a}{1-r}$.

The number $\frac{a}{1-r}$ is called the **limit** of the sum of n terms, as n is increased without bound. Denoting this limit by s, we have:

$$s = \frac{a}{1-r}.$$

The number s is not the sum of *all* the terms of the series, for since the terms of the series never come to an end, the operation of adding them cannot be completed. We cannot end an unending process. The number s is simply the number to which the sum of the first n terms of the series approximates more and more closely as n increases.

For example:

When $a = 4$, and $r = \frac{1}{2}$, then $s = \frac{4}{1-\frac{1}{2}} = 8$.

To test this, we form successive values of s_n.

$$s_1 = 4.$$
$$s_2 = 6.$$
$$s_3 = 7.$$
$$s_4 = 7\tfrac{1}{2}.$$
$$s_5 = 7\tfrac{3}{4}.$$
$$s_6 = 7\tfrac{7}{8}.$$

It appears that the values of s_n approximate more and more closely to 8 as n is increased.

WRITTEN EXERCISES

Find the limit of the sum of the series:

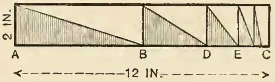
1. $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.
2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.
3. $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} \cdots$.
4. $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \cdots$.

5. Test the results of the preceding exercises by finding successive values of s_n.

6. In an infinite geometric series $s = 2$ and $r = \frac{1}{2}$; find a.

7. Find the fraction which is the limit of .333333 ⋯, or $.3 + .03 + .003 + \cdots$.

8. Find the limit of $.23232323 \cdots$ or $.23 + .0023 + .000023 + \cdots$.



9. Triangles are drawn in a rectangle of dimensions indicated, B being the midpoint of AC, D that of BC, E that of DC, and so on. What limit does the sum of the areas of the triangles approach as more and more triangles are taken?

ADDITIONAL EXERCISES

1. Find the sum of 16 terms of the series,

$$27,\ 22\tfrac{1}{2},\ 18,\ 13\tfrac{1}{2},\ \cdots.$$

2. Find the sum of 18 terms of the series,

$$36,\ 12,\ 4,\ \tfrac{4}{3},\ \cdots.$$

3. The difference between two numbers is 48. The arithmetic mean exceeds the geometric mean by 18. Find the numbers.

4. Express as a geometric series the decimal fraction

$$.0373737\cdots.$$

What is its limiting value?

5. If $\dfrac{1}{b-a}$, $\dfrac{1}{2b}$, $\dfrac{1}{b-c}$, are in arithmetical progression, show that a, b, c are in geometric progression.

Suggestion. The supposition means that

$$\frac{1}{b-a} - \frac{1}{2b} = \frac{1}{2b} - \frac{1}{b-c}.$$

This reduces to $b^2 = ac$.

6. Find the amount in n years of P dollars at r per cent per annum, interest being compounded annually.

7. During a truce, a certain army A loses by sickness 14 men the first day, 15 the second, 16 the third, and so on; while the opposing army B loses 12 men every day. At the end of fifty days the armies are found to be of equal size. Find the difference between the two armies at the beginning of the truce.

8. A strip of carpet one half inch thick and $29\frac{6}{7}$ feet long is rolled on a roller four inches in diameter. Find how many turns there will be, remembering that each turn increases the diameter by one inch, and taking as the length of a circumference $\frac{22}{7}$ times the diameter.

9. Insert between 1 and 21 a series of arithmetic means such that the sum of the last three is 48.

10. If $\frac{a}{b}=\frac{c}{d}$, prove that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

11. The sum of the first ten terms of a geometric series is 244 times the sum of the first five terms; and the sum of the fourth and the sixth term is 135. Find the first term and the common ratio.

CHAPTER XII

ZERO: INTERPRETATION OF RESULTS

ZERO AND ITS PROPERTIES

224. Definition of Zero. Zero may be defined as the result of subtracting a number from itself.

$$6-6=0;\ a-a=0.$$

225. Addition. By definition of zero, $a+0=a+b-b=a$, since to add b and immediately to take it away again leaves the original number a.

226. Subtraction. Similarly, $a-0=a-(b-b)=a-b+b=a$, since to take away b, then at once to replace it, leaves the original number a.

To add or subtract zero does not alter the original number.

227. Multiplication. By definition of zero,

$$0\cdot a=(b-b)a=ba-ba=0.$$

That is, if one factor is zero, the product is zero.

Multiplication by zero simply causes the multiplicand to vanish.

228. Division. We recall that division is the process of finding a number (quotient) which when multiplied by a given number (divisor) shall have a given product (dividend).

$12\div 3$ or $\frac{12}{3}$ simply proposes the problem: By what must 3 be multiplied to produce 12? The proof that $12\div 3=4$ is the fact that $3\times 4=12$.

Likewise $a\div 0$, or $\dfrac{a}{0}$, simply proposes the problem, By what must zero be multiplied to produce a?

Let x denote the desired number. Then $0\cdot x=a$.

But we know that zero times any number is zero. If a is not zero, there is no number x that satisfies the above equation. That is, $\frac{a}{0} =$ no number, since there is no number whose product with 0 is a.

If a is zero, every number x satisfies the equation. That is, $\frac{0}{0} =$ any number, since 0 times any number $= 0$.

Division by zero is therefore either entirely indefinite or impossible. In either case it is not admissible.

229. If we divide one literal expression by another, there is no guarantee that the result is correct for those values of the letters that make the divisor zero.

EXAMPLE

Let	$a = b.$	(1)
Multiplying both members by a,	$a^2 = ab.$	(2)
Subtracting b^2 from both members,	$a^2 - b^2 = ab - b^2.$	(3)
Factoring,	$(a + b)(a - b) = b(a - b).$	(4)
Dividing both members by $a - b$,	$a + b = b.$	(5)
Substituting the value of a from (1),	$b + b = b.$	(6)
Or,	$2b = b.$	(7)
Dividing by b,	$2 = 1.$	(8)

The work is quite correct to equation (4) inclusive. But by dividing equation (4) by an expression that, according to the conditions of the problem, is zero, we find as result an incorrect equation.

230. In all divisions, therefore, we must assure ourselves that the divisor is not zero. If a literal divisor is used, the result can be depended upon only for such values of the letters as do not make the divisor zero.

EXAMPLES

1. The town B is d miles distant from A; two trains leave A and B simultaneously, going in the same direction (that from A towards B), at the rate of m mi. per hour (train from A)

and q mi. per hour (train from B). At what distance from B will the two trains be together?

Solving this problem by the usual method, we find as the result $\dfrac{qd}{m-q}$.

If $d \neq 0$ (read "d is not equal to 0"), and if $m=q$, the result assumes the form $\dfrac{qd}{0}$, an indicated division by zero. This means that the problem is impossible under these conditions. This is evident also from the meaning of m and q in the problem. If the two trains go in the same direction at the same rate, the one will always remain d miles behind the other.

If, however, $d=0$, and $m=q$, the result assumes the form $\dfrac{0}{0}$, which equals any number whatever. This also agrees with the conditions of the problem. If d is zero, B and A are coincident, and the two trains are together at starting. If $m=q$, they both run at the same rate, and always remain together. They are therefore together at every distance from B.

2. We have solved (First Course, Sec. 267, p. 222) the equations

$$ax+by=e,$$
$$cx+dy=f.$$

with the result $\quad x=\dfrac{de-bf}{ad-bc};\ y=\dfrac{af-ec}{ad-bc}.$

I. Let us give the letters a, b, c, d, such values that $ad-bc=0$; for example, $a=2$, $b=1$, $c=4$, $d=2$. And let us give e and f such values that $de-bf$ is not 0; for example, $e=5$, $f=4$.

Then the above results become:

$$x=\frac{6}{0};\ y=-\frac{12}{0}.$$

The indicated division by zero means that the problem is impossible. There is no pair of values that satisfies both equations. This appears readily also by substituting the values of $a \cdots f$ in the given equations, which then become:

$$2x+y=5,$$
$$4x+2y=4.$$

Dividing the second equation by 2, the system becomes:

$$2x+y=5,$$
$$2x+y=2,$$

and it is obvious that no set of values of x and y can make $2x+y$ equal to 5 and also equal to 2.

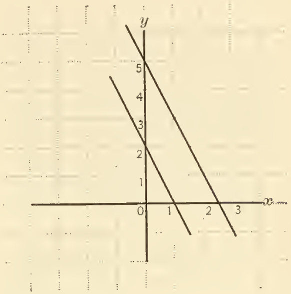
The two equations are called **incompatible** or **contradictory.**

This condition can be illustrated graphically:

Drawing the graphs of

$2x+y=5$ and $2x+y=2$,

the two lines appear to be parallel. That two parallel straight lines do not intersect is the geometric condition corresponding to the fact that a system of two incompatible equations has no solution.



II. Retaining the values of $a \cdots d$ above, let us give e and f such values that the numerators of the result both become zero; for example, $e=5$, $f=10$.

The result assumes the form:

$$x=\frac{0}{0};\ y=\frac{0}{0}.$$

This indicates that x may have any value; and also that y may have any value.

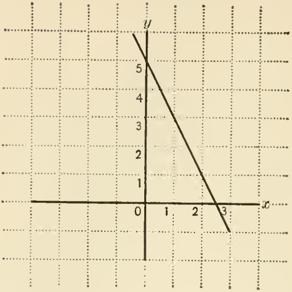
Substituting the values of $a \cdots f$ in the given equations, they become:

$$2x+y=5$$
$$4x+2y=10.$$

It appears that the second equation is twice the first, and hence equivalent to it. Any values of x and y that satisfy the first, will also satisfy the second.

We can choose arbitrarily any value for x and then determine a value of y to go with it by means of the first equation. For example, choosing $x=3$, then $2\cdot 3+y=5$, which gives $y=-1$. These values of x and y satisfy both equations. Similarly, any value can be chosen for y, and a value of x can be determined such that the pair of values satisfies the given system. That is, any value of x is a root; likewise any value of y is a root, in agreement with the meaning of the form, $\frac{0}{0}$, assumed by the result of the general solution.

The two equations are **dependent**. Every solution of one is a solution of the other. The conditions can be illustrated graphically.



If we undertake to make the graphs of the two equations as given, we find that they lead to the same straight line. The two graphs are coincident; every point of the straight line is a common point of the two graphs. Any abscissa x is the abscissa of a common point of the graphs; any ordinate y is the ordinate of a common point of the graphs.

NOTE. The study of expressions which may assume the exceptional forms mentioned above, especially those which may assume the form $\frac{0}{0}$, is very important, both from the point of view of later mathematics and the physical sciences; but what has been said above will suffice for the needs of the present work.

231. We have thus seen that systems of two linear equations in two unknowns may be classified as follows:

1. Independent (the ordinary case, admitting one solution).
2. Contradictory (admitting no solution).
3. Dependent (admitting a boundless number of solutions).

WRITTEN EXERCISES

Construct the graphs of each of the following systems and classify them according to Sec. 231:

1. $3x + y = 2,$
$x + y = 0.$

2. $2x - y = 1,$
$4x - 2y = 2.$

3. $s - t = 6,$
$s + t = 6.$

4. $x + 2z = 10,$
$x + 3z = 11.$

5. $x = 25,$
$y = 10.$

6. $10x + 5y = 25,$
$2x + y = 5.$

7. $7x + 14y = 7,$
$x + 2y = 2.$

8. $12x - 3y = 8,$
$3y - x = 4.$

232. The discussions above are instances of what may be called **interpretation of results**. That is, after the conditions of a problem have been expressed by equations, and the equations solved, the result must be examined to see whether it is admissible under the conditions of the problem; the various possible combinations of the literal expressions given must be discussed; and exceptional or noteworthy sets of values pointed out.

EXAMPLES

1. Find three consecutive integers such that their sum shall be equal to 3 times the second.

SOLUTION. 1. Let $x =$ the first.

2. Then $x + 1 =$ the second,

3. and $x + 2 =$ the third.

4. $\therefore x + (x + 1) + (x + 2) = 3(x + 1)$, by the conditions of the problem.

5. $\therefore (3 - 3)(x + 1) = 0$, or $0(x + 1) = 0$.

INTERPRETATION OF THE RESULT. The equation determines no particular value of x; it exists for every value of x. Consequently, every three consecutive integers must satisfy the given conditions.

2. Find three consecutive integers whose sum is 57, and the sum of the first and third is 40.

SOLUTION. 1. Let $x =$ the first.

2. Then $x + 1 =$ the second,

3. and $x + 2 =$ the third.

4. Then, $x + (x + 1) + (x + 2) = 57$,

5. and $x + (x + 2) = 40$, by the given conditions.

6. From (4), $x = 18$.

INTERPRETATION OF THE RESULT. This result for x will not satisfy equation (5); therefore no three consecutive integers satisfy the conditions of the problem.

WRITTEN EXERCISES

Solve and interpret the results:

1. Fifteen clerks receive together \$150 per week; some receive \$8 and others \$12 per week. How many are there receiving each salary?

2. A train starts from New York to Richmond via Philadelphia and Baltimore at the rate of 30 miles an hour, and two hours later another train starts from Philadelphia for Richmond at the rate of 20 miles an hour. How far beyond Baltimore will the first train overtake the second, given that the distance from New York to Philadelphia is 90 miles and from Philadelphia to Baltimore 96 miles?

3. The hot-water faucet of a bath tub will fill it in 14 minutes, the cold-water faucet in 10 minutes, and the waste pipe will empty it in 4 minutes. How long will it take to fill the tub when both faucets and the waste pipe are opened?

4. If the freight on a certain class of goods is 2 cents per ton per mile, together with a fixed charge of 5 cents per ton for loading, how far can 2000 tons be sent for $80?

5. Find three consecutive integers whose sum equals the product of the first and the last.

SUPPLEMENT

GEOMETRIC PROBLEMS FOR ALGEBRAIC SOLUTION

The problems in the following list may be used as supplementary work for pupils that have studied plane geometry. In the body of the Algebra numerous problems have been given applying geometric facts which the pupil has learned in the study of mensuration in arithmetic. In the following list the problems contain the application of other relations and theorems of geometry, and typical solutions have been inserted to suggest to the pupil the method of attack.

LINEAR EQUATIONS. ONE UNKNOWN

1. In a given triangle one angle is twice another, and the third angle is 24°. Find the unknown angles.

SOLUTION. Let x be one of the unknown angles,

then $2x$ is the other. *(1)*

Because the sum of the angles of a triangle = 180°,

$$x + 2x + 24° = 180°. \quad (2)$$

Solving equation (2),

$$x = 52° \text{ and } 2x = 104°. \quad (3)$$

The angles of the triangle are 52°, 104°, 24°.

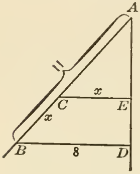
2. In a certain triangle one angle is three times another, and the third angle is 36°. Find the unknown angles.

3. In a given right-angled triangle one acute angle is $\frac{2}{3}$ the other. Find the angles.

4. In a certain isosceles triangle the angle opposite to the base is 18°. Find the angles at the base.

5. The three angles A, B, and C of a given triangle are in the ratio of 2, 3, and 5. Find the angles.

6. Given an angle A such that a point B situated on one side 11 in. from the vertex is 8 in. distant from the other side. Find a point C on the same side as B, and equidistant from B and the other side of the angle.



SOLUTION. Let A be the given angle, then the figure represents the conditions of the problem.

From the similar triangles ACE and ABD, we have

$$\frac{AC}{CE}=\frac{AB}{BD},$$

or,
$$\frac{11-x}{x}=\frac{11}{8}. \qquad (1)$$

From (1),
$$88-8x=11x. \qquad (2)$$

Then,
$$88=19x, \qquad (3)$$

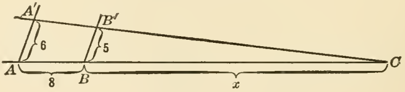
and
$$x=\tfrac{88}{19}. \qquad (4)$$

The distance CB is $\frac{88}{19}$ in., or $4\frac{12}{19}$.

7. Solve problem 6, if the point B is 9 in. from A and 6 in. from side AD.

8. Solve problem 6, if the point B is a in. from the vertex of the angle and b in. from the other side of the angle.

9. Two points A and B are 8 in. apart. Parallels are drawn through A and B; on these parallels the points A' and B' are located on the same side of the straight line through AB and at distances 6 in. and 5 in. from A and B, respectively. Determine the point where the line $A'B'$ cuts the line AB.



SOLUTION. Let C be the desired point and let $BC=x$.

Then, by similar triangles,

$$\frac{BC}{BB'}=\frac{AC}{AA'}, \qquad (1)$$

or
$$\frac{x}{5}=\frac{x+8}{6}. \qquad (2)$$

Hence, $$6x = 5x + 40. \qquad (3)$$
$$x = 40. \qquad (4)$$

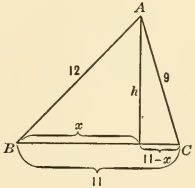
The point is 40 in. from B.

10. Solve the same problem if A' and B' lie on opposite sides of AB.

11. Solve the same problem if the distance AB is d, and the points A', B' lie on the same side of AB, and at distances a and b from A and B respectively, with $a > b$.

12. Solve the preceding problem if the points A' and B' lie on opposite sides of AB.

13. The three sides of a triangle are 11, 9, 12. A perpendicular is dropped on the side of length 11 from the opposite vertex. Find the lengths of the segments into which the foot of the perpendicular divides that side.



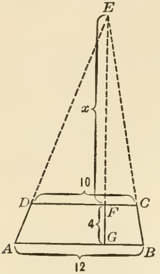
Solution. Using the notations of the figure,
$$h^2 = 12^2 - x^2, \qquad (1)$$
and
$$h^2 = 9^2 - (11 - x)^2. \qquad (2)$$
From (1) and (2), $$12^2 - x^2 = 9^2 - (11 - x)^2. \qquad (3)$$
Rearranging (3), $$12^2 - 9^2 + 11^2 = 22x. \qquad (4)$$
Solving (4), $$x = \tfrac{92}{11}. \qquad (5)$$
The other segment is $11 - x$, or $\frac{29}{11}$. (6)

The segments are $\frac{92}{11}$ and $\frac{29}{11}$.

14. Solve the preceding problem if the sides are 4, 7, 9, and the perpendicular is dropped on the side of length 7.

15. Solve the same problem if the sides of the triangle are a, b, c, and the perpendicular is dropped on the side of length a.

16. The lower base of a trapezoid is 12, the upper base is 10, and the altitude is 4. Determine the altitude of the triangle formed by the upper base and the prolongation of the two non-parallel sides until they meet.



SOLUTION. Using the notations of the figure,

$$\frac{EF}{EG}=\frac{DC}{AB}, \tag{1}$$

or

$$\frac{x}{x+4}=\frac{10}{12}. \tag{2}$$

Hence,

$$12\,x=10\,x+40, \tag{3}$$

or

$$x=20. \tag{4}$$

The altitude is 20.

17. Solve the same problem if the lower base is a, the upper base b, and the altitude h.

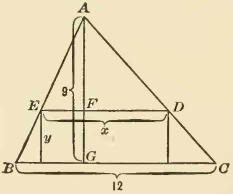
LINEAR EQUATIONS. TWO UNKNOWNS

18. A rectangle 5 in. longer than it is wide is inscribed in a triangle of base 12 in. and altitude 9 in., the longer side resting on the base of the triangle. Find the dimensions of the rectangle.

SOLUTION. Let x denote the longer side and y the shorter.

Then,

$$x-y=5. \tag{1}$$



In the similar triangles ABC and AED,

$$\frac{AF}{AG}=\frac{ED}{BC}, \tag{2}$$

or

$$\frac{9-y}{9}=\frac{x}{12}. \tag{3}$$

From (3), $108 - 12y = 9x$. (4)

From (1) and (4), $108 - 12y = 9(y + 5)$, (5)

or $21y = 63$. (6)

$y = 3$. (7)

From (7) and (1), $x = 8$. (8)

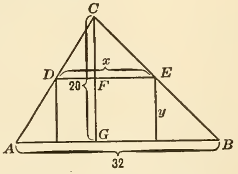
The dimensions of the rectangle are 3 in. and 8 in.

19. Solve the preceding problem if the shorter side rests on the base of length 12 in.

20. Solve Problem **18** if the difference of the sides is d, the length of the base of the triangle is a, the altitude is h, and the longer side of the rectangle rests on the base of the triangle.

21. Solve the preceding problem if the shorter side of the rectangle rests on the given base of the triangle.

22. A rectangle similar to a rectangle whose sides are 5 and 8 is inscribed in a triangle of base 32 and altitude 20. The longer side of the rectangle rests on the given base of the triangle. Find the dimensions of the rectangle.



Solution. Let x and y denote the sides of the inscribed rectangle. Then from the similar triangles EDC and ABC,

$$\frac{DE}{AB} = \frac{CF}{CG}, \tag{1}$$

or

$$\frac{x}{32} = \frac{20 - y}{20}. \tag{2}$$

From the similarity of the rectangles,

$$\frac{x}{y} = \frac{8}{5}. \tag{3}$$

From (3), $x = \frac{8}{5}y$. (4)

From (2), $20x = 640 - 32y$. (5)

From (4) and (5), $32y = 640 - 32y$, (6)

or $64y = 640$. (7)

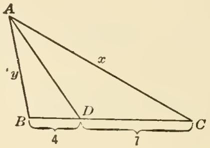
$y = 10$. (8)

From (8) and (4), $x = 16$. (9)

23. Solve the preceding problem if the shorter side of the rectangle rests on the given side of the triangle.

24. Solve the same problem if the given side of the triangle is of length a, and the altitude on it is of length h, and the given rectangle has dimensions l and m, provided the inscribed rectangle has its side corresponding to the side l of the given rectangle resting on the given base of the triangle.

25. The bisector of an angle of a given triangle divides the side opposite to the angle into two segments of lengths 4 in. and 7 in. The difference between the other two sides of the triangle is 5 in. Find the perimeter of the triangle.



SOLUTION. Let ABC be the given triangle, AD the bisector of angle A, and x and y the required sides.

Then, $\quad x - y = 5$, given in the problem, $\quad (1)$

and $\frac{x}{y} = \frac{7}{4}$, by geometry the bisector divides the opposite side into segments proportional to the adjacent sides. $\quad (2)$

$$4x = 7y, \text{ from } (2). \quad (3)$$

$$7y - 4y = 20, \text{ from } (3) \text{ and } 4 \text{ times } (1). \quad (4)$$

Then, $\quad y = 6\frac{2}{3}$, solving (4). $\quad (5)$

Then, $\quad x = 11\frac{2}{3}$, from (5) and (1). $\quad (6)$

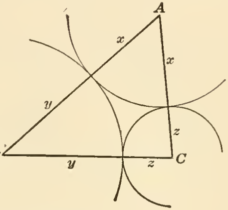
The perimeter is 11 in. + $6\frac{2}{3}$ in. + $11\frac{2}{3}$ in. = $29\frac{1}{3}$ in.

26. Solve the preceding problem, if the segments of the base are l and m and the difference between the sides is d.

27. The sides of a triangle are 8 ft., 12 ft., and 15 ft., and the angle between the sides 8 and 12 is bisected by a line cutting the side 15. What is the length of each segment of the line 15?

LINEAR EQUATIONS. THREE UNKNOWNS

28. The points A, B, and C are situated so that $AB = 8$ in., $BC = 6$ in., $AC = 5$ in. Find the radii of three circles having the three points as centers and each tangent to the other two externally.



SOLUTION. Let x, y, z, be the radii of the three circles as indicated in the figure.

Then,
$$x + y = 8, \quad (1)$$
$$x + z = 5, \quad (2)$$
$$y + z = 6. \quad (3)$$

Adding (1), (2), and (3), and dividing the result by 2,
$$x + y + z = \tfrac{19}{2}. \quad (4)$$

From (3) and (4), $\quad x = \frac{7}{2}. \quad (5)$

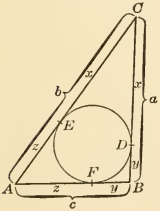
From (2) and (4), $\quad y = \frac{9}{2}. \quad (6)$

From (1) and (4), $\quad z = \frac{3}{2}. \quad (7)$

The radii are $3\frac{1}{2}$ in., $4\frac{1}{2}$ in., and $1\frac{1}{2}$ in.

29. Solve the same problem if $AB = 12$, $BC = 16$, and $AC = 20$.

30. Solve Problem 28, if $BC = a$, $AC = b$, $AB = c$.



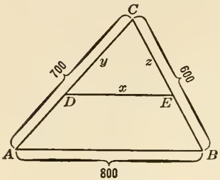
31. A triangle ABC is circumscribed about a circle and its sides are tangent at points E, D, F. The sides of the triangle are $a = 8$ in., $b = 15$ in., and $c = 12$ in. Find the segments into which points E, D, F divide the sides.

SOLUTION. The tangents from an external point are equal, hence in the figure,
$$x + y = 8, \quad (1)$$
$$x + z = 15, \quad (2)$$
$$y + z = 12. \quad (3)$$

Solving these equations, the segments are $x = ?\quad y = ?\quad z = ?$

32. Solve the same problem, if $a = 24$ ft., $b = 10$ ft., and $c = 24$ ft.

33. A man owned a triangular, unfenced field of sides 600 yd., 700 yd., and 800 yd. He sold a triangular piece cut off by a straight line parallel to the side of length 800 yd. and found that 1800 running yards of fence would be required to inclose what remained. Find the lengths of the sides of the portion sold.



Solution. Using the notations of the figure,

$$AB + BC + AC + 2\,DE = AB + BE + EC + CD + DA + 2\,DE$$

$$= (AB + BE + ED + DA) + (EC + CD + DE). \quad (1)$$

Hence, $$2100 + 2x = 1800 + (x + y + z). \quad (2)$$

But $$\frac{CD}{CA} = \frac{DE}{AB}, \quad (3)$$

and $$\frac{CE}{CB} = \frac{DE}{AB}, \quad (4)$$

or $$\frac{y}{700} = \frac{x}{800}, \quad (5)$$

and $$\frac{z}{600} = \frac{x}{800}. \quad (6)$$

Hence, $$y = \frac{7x}{8}, \quad (7)$$

$$z = \frac{3x}{4}. \quad (8)$$

From (7) and (8), $$x + y + z = \frac{21x}{8}. \quad (9)$$

From (9) and (2), $$2100 + 2x = 1800 + \frac{21x}{8}. \quad (10)$$

Hence, $$\frac{5x}{8} = 300. \quad (11)$$

$$x = 480. \quad (12)$$

From (12) and (7), $$y = 420. \quad (13)$$

From (12) and (8), $$z = 360. \quad (14)$$

The lengths of the sides are: $DE = 480$ yd., $EC = 360$ yd., and $CD = 420$ yd.

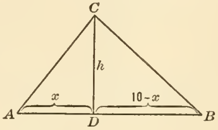
34. Solve Problem 33 if the division line runs parallel to the side of length 700 yd.

35. Solve Problem 33 if the division line runs parallel to the side of length 600 yd.

36. Solve Problem 33 if the lengths of the sides are a, b, c, with the division line parallel to a, and the length of fence required is $2p$.

QUADRATIC EQUATIONS

37. The sides of a triangle are $AC = 7$, $BC = 9$, and $AB = 10$. Calculate the length of the altitude on the side 10, and of the two segments into which the altitude divides that side.



SOLUTION. Using the notations of the figure,

$$h^2 = 7^2 - x^2, \tag{1}$$

and

$$h^2 = 9^2 - (10 - x)^2. \tag{2}$$

Then,

$$7^2 - x^2 = 9^2 - (10 - x)^2, \tag{3}$$

or

$$7^2 = 9^2 - 100 + 20x. \tag{4}$$

Then,

$$68 = 20x, \tag{5}$$

and

$$x = \tfrac{17}{5}; \tag{6}$$

also,

$$10 - x = \tfrac{33}{5}. \tag{7}$$

By (1),

$$h^2 = 7^2 - (\tfrac{17}{5})^2. \tag{8}$$

Then,

$$h = \frac{\sqrt{7^2 \cdot 5^2 - 17^2}}{5}$$

$$= \frac{\sqrt{(35 + 17)(35 - 17)}}{5}$$

$$= \frac{\sqrt{52 \cdot 18}}{5}$$

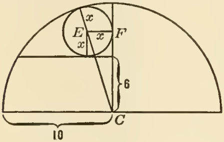
$$= \frac{\sqrt{2 \cdot 26 \cdot 2 \cdot 9}}{5} = \frac{6\sqrt{26}}{5}. \tag{9}$$

The segments are $\frac{17}{5}$ and $\frac{33}{5}$ and the altitude is $\frac{6\sqrt{26}}{5}$.

38. Calculate similarly the length of the altitude on the side of length 7, and of the segments into which the altitude divides that side.

39. Calculate similarly the length of the altitude on the side of length 9, and of the segments into which the altitude divides that side.

40. If the lengths of the sides of a triangle are a, b, c, calculate the lengths of the segments into which each side is divided by the altitude on that side.



41. In a circle of radius 10, a chord is drawn at distance 6 from the center. Find the radius of a circle that is tangent to the circle, to the chord, and to a diameter perpendicular to it.

SOLUTION. Using the notations of the figure:

$$\overline{EC}^2 = \overline{EF}^2 + \overline{FC}^2. \qquad (1)$$

$$(10 - x)^2 = x^2 + (6 + x)^2. \qquad (2)$$

$$100 - 20x + x^2 = x^2 + 36 + 12x + x^2. \qquad (3)$$

$$x^2 + 32x - 64 = 0. \qquad (4)$$

$$x = -\frac{32 \pm \sqrt{32^2 + 4 \cdot 64}}{2} \qquad (5)$$

$$= -\frac{32 \pm 16\sqrt{4 + 1}}{2} \qquad (6)$$

$$= -16 \pm 8\sqrt{5}. \qquad (7)$$

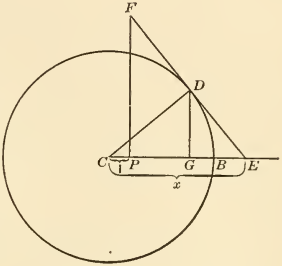
The negative value of x being inadmissible under the geometric conditions, we have:

$$x = -16 + 8\sqrt{5}. \qquad (8)$$

42. Solve the same problem, if the radius of the given circle is 12 ft. and the chord is 4 ft. from the center.

43. Solve Problem 41, if the radius of the given circle is r and the chord is at d distance from the center.

44. A point P is selected on a diameter of a circle of radius 6, at the distance 1 from the center. At P a perpendicular is erected to the diameter in question, and a tangent is drawn to the circle such that the point of contact of the tangent bisects the segment of the tangent lying between the perpendicular and the diameter produced. Find the distance from the center to the point where the tangent cuts the diameter produced.



SOLUTION. Let $$CE = x. \tag{1}$$

Then in the right triangle CDE,

$$\overline{DE}^2 = x^2 - 6^2. \tag{2}$$

From the similar triangles DCE and DGE,

$$\frac{DE}{CE} = \frac{GE}{DE}, \tag{3}$$

or

$$\overline{DE}^2 = CE \cdot GE \tag{4}$$

$$= x \cdot GE. \tag{5}$$

Since D bisects FE, $$GE = \frac{PE}{2} \tag{6}$$

$$= \frac{x-1}{2}. \tag{7}$$

From (2), (5), and (7), $$\frac{x(x-1)}{2} = x^2 - 6^2, \tag{8}$$

or

$$x^2 - x = 2x^2 - 72. \tag{9}$$

$$\therefore\ x^2 + x - 72 = 0. \tag{10}$$

$$x = -\frac{1 \pm \sqrt{1+288}}{2} \tag{11}$$

$$= -9,\ 8. \tag{12}$$

The negative root also indicates a solution. It means that a second tangent satisfying the required conditions cuts the diameter produced on the opposite side from P, at the distance 9.

45. Solve the same problem if the radius is 12 and the point lies at the distance 2 from the center.

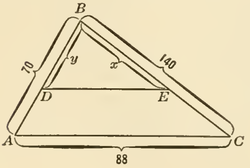
46. Solve the same problem if the radius is 15, and the distance of P from the center is 35.

47. Solve the same problem if the radius is r and the distance from the center is d.

48. The tangent to a circle is a mean proportional between the segments of the secant from the same point. Find the length of the tangent, if the segments of the secant are 4 ft. and 9 ft.

49. Two chords AB and CD intersect at O within the circle. The product of OA and OB equals the product of OC and OD. Given $OA = 4$, $OB = 8$, and $CD = 12$, find OC and OD.

50. The owner of a triangular lot whose sides are 70, 88, and 140 rd. in length wishes to divide it by a straight fence into two parts that shall be equal in area and also have the same perimeter. If the fence connects the sides of length 70 and 140 rd., how must it be placed?



SOLUTION. Let DE be the desired position of the fence.

Then, $$\triangle ABC = 2\triangle DBE. \tag{1}$$

Since the triangles have one angle in common,

$$\frac{\triangle ABC}{\triangle DBE} = \frac{70 \cdot 140}{xy}, \tag{2}$$

or, by (1), $$2 = \frac{70 \cdot 140}{xy}, \tag{3}$$

or $$xy = 35 \cdot 140. \tag{4}$$

By the conditions of the problem,

$$BD + BE + DE = DA + AC + CE + ED. \tag{5}$$

Subtracting DE from both members and replacing the other lines by their values,

$$x + y = 70 - y + 140 - x + 88, \quad (6)$$

$$2(x + y) = 298, \quad (7)$$

$$x + y = 149. \quad (8)$$

From (4),
$$4\,xy = 4 \cdot 35 \cdot 140$$
$$= \overline{140}^2. \quad (9)$$

Squaring (8) and subtracting (9) from the result,

$$(x - y)^2 = \overline{149}^2 - \overline{140}^2 \quad (10)$$
$$= (149 + 140)(149 - 140) \quad (11)$$
$$= 289 \cdot 9. \quad (12)$$

Then,
$$x - y = \pm 17 \cdot 3$$
$$= \pm 51. \quad (13)$$

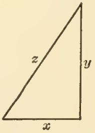
From (8) and (13),
$$x = 100,\ 49. \quad (14)$$
$$y = 49,\ 100. \quad (15)$$

These values satisfy the algebraic equations, but in the concrete problem $y = 100$ is inadmissible, since y lies on the side of length 70. Hence, in the concrete problem, the result is $x = 100$, $y = 49$.

51. Solve the same problem if the fence connects the sides of length 70 and 88.

52. Solve the same problem if the fence connects the sides of length 88 and 140.

53. Solve the same problem if the sides are of length a, b, c, and the fence connects the sides of lengths a and c.



54. Find the sides of a right-angled triangle, given its area 25, and its perimeter 30.

Solution. Let x, y, z denote the sides of the triangle, z being the hypotenuse.

Then,
$$x^2 + y^2 = z^2, \quad (1)$$
$$x + y + z = 30, \quad (2)$$

and
$$\frac{xy}{2} = 25. \quad (3)$$

Multiplying both members of (3) by 4,

$$2\,xy = 100. \quad (4)$$

Adding (4) and (1),
$$x^2 + 2\,xy + y^2 = z^2 + 100, \quad (5)$$

or
$$(x + y)^2 = z^2 + 100. \quad (6)$$

From (2),
$$x + y = 30 - z, \quad (7)$$

or
$$(x + y)^2 = (30 - z)^2. \quad (8)$$

From (8) and (6),
$$(30 - z)^2 = z^2 + 100. \tag{9}$$
$$900 - 60z + z^2 = z^2 + 100. \tag{10}$$
$$60z = 800. \tag{11}$$
$$z = \tfrac{40}{3}. \tag{12}$$
From (7),
$$x + y = \tfrac{50}{3}, \tag{13}$$
or
$$x^2 + 2xy + y^2 = \tfrac{2500}{9}. \tag{14}$$

Multiplying (4) by 2 and subtracting the result from (14),
$$x^2 - 2xy + y^2 = \tfrac{700}{9}, \tag{15}$$
or
$$x - y = \pm\frac{10\sqrt{7}}{3}. \tag{16}$$

Adding (13) and (16) and dividing the result by 2,
$$x = \frac{25 \pm 5\sqrt{7}}{3}. \tag{17}$$

Subtracting (16) from (13) and dividing the result by 2,
$$y = \frac{25 \mp 5\sqrt{7}}{3}. \tag{18}$$

The sides are $\dfrac{25 + 5\sqrt{7}}{3}$, $\dfrac{25 - 5\sqrt{7}}{3}$, and $\dfrac{40}{3}$.

55. Solve the same problem if the area of the triangle is 64, and the perimeter 48.

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